Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 9 November 2021 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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3 pages

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 25]

In this question you will explore some of the properties of special functions f and g and their relationship with the trigonometric functions, sine and cosine.

Functions f and g are defined as $f(z)=\dfrac{\mathrm{e}^{z}+\mathrm{e}^{-z}}{2}$ and $g(z)=\dfrac{\mathrm{e}^{z}-\mathrm{e}^{-z}}{2}$, where $z \in \mathbb{C}$.

Consider t and u, such that $t,\ u \in \mathbb{R}$.

(a) Verify that $u=f(t)$ satisfies the differential equation $\dfrac{\mathrm{d}^2u}{\mathrm{d}t^2}=u$. [2]

(b) Show that $\left(f(t)\right)^2+\left(g(t)\right)^2=f(2t)$. [3]

(c) Using $\mathrm{e}^{\mathrm{i}u}=\cos u+\mathrm{i}\sin u$, find expressions, in terms of $\sin u$ and $\cos u$, for

(i) $f(\mathrm{i}u)$; [3]

(ii) $g(\mathrm{i}u)$. [2]

(d) Hence find, and simplify, an expression for $\left(f(\mathrm{i}u)\right)^2+\left(g(\mathrm{i}u)\right)^2$. [2]

(e) Show that $\left(f(t)\right)^2-\left(g(t)\right)^2=\left(f(\mathrm{i}u)\right)^2-\left(g(\mathrm{i}u)\right)^2$. [4]

The functions $\cos x$ and $\sin x$ are known as circular functions as the general point $(\cos\theta, \sin\theta)$ defines points on the unit circle with equation $x^2+y^2=1$.

The functions $f(x)$ and $g(x)$ are known as hyperbolic functions, as the general point $\left(f(\theta), g(\theta)\right)$ defines points on a curve known as a hyperbola with equation $x^2-y^2=1$. This hyperbola has two asymptotes.

(f) Sketch the graph of $x^2-y^2=1$, stating the coordinates of any axis intercepts and the equation of each asymptote. [4]

The hyperbola with equation $x^2-y^2=1$ can be rotated to coincide with the curve defined by $xy=k,\ k\in\mathbb{R}$.

(g) Find the possible values of k. [5]

2. [Maximum mark: 30]

In this question you will be exploring the strategies required to solve a system of linear differential equations.

Consider the system of linear differential equations of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - y \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = ax + y,$$

where $x, y, t \in \mathbb{R}^+$ and a is a parameter.

First consider the case where $a = 0$.

(a) (i) By solving the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = y$, show that $y = A\mathrm{e}^t$ where A is a constant. [3]

(ii) Show that $\frac{\mathrm{d}x}{\mathrm{d}t} - x = -A\mathrm{e}^t$. [1]

(iii) Solve the differential equation in part (a)(ii) to find x as a function of t. [4]

Now consider the case where $a = -1$.

(b) (i) By differentiating $\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y$ with respect to t, show that $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 2\frac{\mathrm{d}y}{\mathrm{d}t}$. [3]

(ii) By substituting $Y = \frac{\mathrm{d}y}{\mathrm{d}t}$, show that $Y = B\mathrm{e}^{2t}$ where B is a constant. [3]

(iii) Hence find y as a function of t. [2]

(iv) Hence show that $x = -\frac{B}{2}\mathrm{e}^{2t} + C$, where C is a constant. [3]

Now consider the case where $a = -4$.

(c) (i) Show that $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$. [3]

From previous cases, we might conjecture that a solution to this differential equation is $y = F\mathrm{e}^{\lambda t}$, $\lambda \in \mathbb{R}$ and F is a constant.

(ii) Find the two values for λ that satisfy $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$. [4]

Let the two values found in part (c)(ii) be λ_1 and λ_2.

(iii) Verify that $y = F\mathrm{e}^{\lambda_1 t} + G\mathrm{e}^{\lambda_2 t}$ is a solution to the differential equation in (c)(i), where G is a constant. [4]

References: