M21/5/MATHX/HP1/ENG/TZ2/XX/M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2021

Mathematics: analysis and approaches

Higher level

Paper 1

29 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.
A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
R Marks awarded for clear **Reasoning**.
AG Answer given in the question and so no marks are awarded.
FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1{,}9$ or 1000 and $1{,}000$ and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x} \times \mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x} \times \mathrm{e}^{3x} - \mathrm{e}^{4x} \times \mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. attempt to subtract squares of integers ***(M1)***

$(n+1)^2 - n^2$

EITHER

correct order of subtraction and correct expansion of $(n+1)^2$, seen anywhere ***A1A1***

$= n^2 + 2n + 1 - n^2 \left(= 2n+1\right)$

OR

correct order of subtraction and correct factorization of difference of squares ***A1A1***

$= (n+1-n)(n+1+n)\left(= 2n+1\right)$

THEN

$= n + n + 1$ =RHS ***A1***

Note: Do not award final ***A1*** unless all previous working is correct.

which is the sum of n and $n+1$ ***AG***

Note: If expansion and order of subtraction are correct, award full marks for candidates who find the sum of the integers as $2n+1$ and then show that the difference of the squares (subtracted in the correct order) is $2n+1$.

[4 marks]

2. (a) attempt to use $\cos^2 x = 1 - \sin^2 x$ ***M1***

$2\sin^2 x - 5\sin x + 2 = 0$ ***A1***

EITHER

attempting to factorise ***M1***

$(2\sin x - 1)(\sin x - 2)$ ***A1***

OR

attempting to use the quadratic formula ***M1***

$\sin x = \dfrac{5 \pm \sqrt{5^2 - 4\times 2\times 2}}{4} \left(= \dfrac{5\pm 3}{4}\right)$ ***A1***

THEN

$\sin x = \dfrac{1}{2}$ ***(A1)***

$x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ ***A1A1***

Total [7 marks]

3. **EITHER**

attempt to use the binomial expansion of $(x+k)^7$ ***(M1)***

${}^7C_0x^7k^0 + {}^7C_1x^6k^1 + {}^7C_2x^5k^2 + \ldots$ (or ${}^7C_0k^7x^0 + {}^7C_1k^5x^1 + {}^7C_2k^5x^2 + \ldots$)

identifying the correct term ${}^7C_2x^5k^2$ (or ${}^7C_5k^2x^5$) ***(A1)***

OR

attempt to use the general term ${}^7C_rx^rk^{7-r}$ (or ${}^7C_rk^rx^{7-r}$) ***(M1)***

$r = 2$ (or $r = 5$) ***(A1)***

THEN

${}^7C_2 = 21$ (or ${}^7C_5 = 21$) (seen anywhere) ***(A1)***

$21x^5k^2 = 63x^5$ ($21k^2 = 63$, $k^2 = 3$) ***A1***

$k = \pm\sqrt{3}$ ***A1***

Note: If working shown, award ***M1A1A1A1A0*** for $k = \sqrt{3}$.

[5 marks]

4. (a) $\ln\left(x^2-16\right)=0$ ***(M1)***

$e^0=x^2-16(=1)$

$x^2=17$ OR $x=\pm\sqrt{17}$ ***(A1)***

$a=\sqrt{17}$ ***A1***

[3 marks]

(b) attempt to differentiate (must include $2x$ and/or $\frac{1}{x^2-16}$) ***(M1)***

$f'(x)=\frac{2x}{x^2-16}$ ***A1***

setting their derivative $=\frac{1}{3}$ ***M1***

$\frac{2x}{x^2-16}=\frac{1}{3}$

$x^2-16=6x$ ***OR*** $x^2-6x-16=0$ (or equivalent) ***A1***

valid attempt to solve their quadratic ***(M1)***

$x=8$ ***A1***

Note:	Award ***A0*** if the candidate's final answer includes additional solutions (such as $x=-2,\ 8$).

[6 marks]
Total [9 marks]

5. **METHOD 1**

use of $|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$ on the LHS ***(M1)***

$|\boldsymbol{a}\times\boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$ ***A1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2\left(1-\cos^2\theta\right)$ ***M1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2\theta$ OR $= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - \left(|\boldsymbol{a}||\boldsymbol{b}|\cos\theta\right)^2$ ***A1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - \left(\boldsymbol{a}\cdot\boldsymbol{b}\right)^2$ ***AG***

METHOD 2

use of $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ on the RHS ***(M1)***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2\theta$ ***A1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2\left(1-\cos^2\theta\right)$ ***M1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$ OR $= \left(|\boldsymbol{a}||\boldsymbol{b}|\sin\theta\right)^2$ ***A1***

$= |\boldsymbol{a}\times\boldsymbol{b}|^2$ ***AG***

Note: If candidates attempt this question using cartesian vectors, e.g
$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ **and** $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ **,**
award full marks if fully developed solutions are seen.
Otherwise award no marks.

[4 marks]

6. METHOD 1

attempt to use the cosine rule to find the value of x ***(M1)***

$100 = x^2 + 4x^2 - 2(x)(2x)\left(\frac{3}{4}\right)$ ***A1***

$2x^2 = 100$

$x^2 = 50$ OR $x = \sqrt{50}$ $(= 5\sqrt{2})$ ***A1***

attempt to find $\sin\hat{C}$ (seen anywhere) ***(M1)***

$\sin^2\hat{C} + \left(\frac{3}{4}\right)^2 = 1$ or $x^2 + 3^2 = 4^2$ OR right triangle with side 3 and hypotenuse 4

$\sin\hat{C} = \frac{\sqrt{7}}{4}$ ***(A1)***

Note: The marks for finding $\sin\hat{C}$ may be awarded independently of the first three marks for finding x.

correct substitution into the area formula using their value of x (or x^2) and their value of $\sin\hat{C}$ ***(M1)***

$A = \frac{1}{2}\times 5\sqrt{2}\times 10\sqrt{2}\times\frac{\sqrt{7}}{4}$ or $A = \frac{1}{2}\times\sqrt{50}\times 2\sqrt{50}\times\frac{\sqrt{7}}{4}$

$A = \frac{25\sqrt{7}}{2}$ ***A1***

continued…

METHOD 2

attempt to find the height, h, of the triangle in terms of x ***(M1)***

$h^2+\left(\frac{3}{4}x\right)^2=x^2$ OR $h^2+\left(\frac{5}{4}x\right)^2=10^2$ OR $h=\frac{\sqrt{7}}{4}x$ ***A1***

equating their expressions for either h^2 or h ***(M1)***

$x^2-\left(\frac{3}{4}x\right)^2=10^2-\left(\frac{5}{4}x\right)^2$ OR $\sqrt{100-\frac{25}{16}x^2}=\frac{\sqrt{7}}{4}x$ (or equivalent) ***A1***

$x^2=50$ OR $x=\sqrt{50}$ $\left(=5\sqrt{2}\right)$ ***A1***

correct substitution into the area formula using their value of x (or x^2) ***(M1)***

$A=\frac{1}{2}\times 2\sqrt{50}\times\frac{\sqrt{7}}{4}\sqrt{50}$ OR $A=\frac{1}{2}\left(2\times 5\sqrt{2}\right)\left(\frac{\sqrt{7}}{4}5\sqrt{2}\right)$

$A=\frac{25\sqrt{7}}{2}$ ***A1***

Total [7 marks]

7. $\alpha+\beta+\alpha+\beta=k$ ***(A1)***

$\alpha+\beta=\frac{k}{2}$

$\alpha\beta\left(\alpha+\beta\right)=-3k$ ***(A1)***

$\left(-\frac{k^2}{4}\right)\left(\frac{k}{2}\right)=-3k\ \left(-\frac{k^3}{8}=-3k\right)$ ***M1***

attempting to solve $-\frac{k^3}{8}+3k=0$ (or equivalent) for k ***(M1)***

$k=2\sqrt{6}\ \left(=\sqrt{24}\right)\left(k>0\right)$ ***A1***

Note: Award ***A0*** for $k=\pm2\sqrt{6}\ \left(\pm\sqrt{24}\right)$.

[5 marks]

8. (a) **METHOD 1**

setting at least two components of l_1 and l_2 equal ***M1***

$$3+2\lambda=2+\mu \quad (1)$$
$$2-2\lambda=-\mu \quad (2)$$
$$-1+2\lambda=4+\mu \quad (3)$$

attempt to solve two of the equations eg. adding (1) and (2) ***M1***

gives a contradiction (no solution), eg $5=2$ ***R1***

so l_1 and l_2 do not intersect ***AG***

Note: For an error within the equations award ***M0M1R0.***   **Note:** The contradiction must be correct to award the ***R1.***

[3 marks]

METHOD 2

l_1 and l_2 are parallel, so l_1 and l_2 are either identical or distinct. ***R1***

Attempt to subtract two position vectors from each line,

e.g. $\begin{pmatrix}3\\2\\-1\end{pmatrix}-\begin{pmatrix}2\\0\\4\end{pmatrix}\left(=\begin{pmatrix}1\\2\\-5\end{pmatrix}\right)$ ***M1***

$\begin{pmatrix}3\\2\\-1\end{pmatrix}\neq k\begin{pmatrix}1\\-1\\1\end{pmatrix}$ ***A1***

[3 marks]

(b) **METHOD 1**

l_1 and l_2 are parallel (as $\begin{pmatrix}2\\-2\\2\end{pmatrix}$ is a multiple of $\begin{pmatrix}1\\-1\\1\end{pmatrix}$)

let A be $(3,2,-1)$ on l_1 and let B be $(2,0,4)$ on l_2

Attempt to find vector $\overrightarrow{\mathrm{AB}}\left(=\begin{pmatrix}-1\\-2\\5\end{pmatrix}\right)$ ***(M1)***

Distance required is $\dfrac{\left|\boldsymbol{v}\times\overrightarrow{\mathrm{AB}}\right|}{|\boldsymbol{v}|}$ ***M1***

continued…

Question 8 continued

$= \frac{1}{\sqrt{3}}\left|\begin{pmatrix}1\\-1\\1\end{pmatrix}\times\begin{pmatrix}-1\\-2\\5\end{pmatrix}\right|$ ***(A1)***

$= \frac{1}{\sqrt{3}}\left|\begin{pmatrix}3\\6\\3\end{pmatrix}\right|$ ***A1***

minimum distance is $\sqrt{18}\left(=3\sqrt{2}\right)$ ***A1***

[5 marks]

METHOD 2

l_1 and l_2 are parallel (as $\begin{pmatrix}2\\-2\\2\end{pmatrix}$ is a multiple of $\begin{pmatrix}1\\-1\\1\end{pmatrix}$)

let A be a fixed point on l_1 eg $(3,2,-1)$ and let B be a general point on l_2 $(2+\mu,-\mu,4+\mu)$

attempt to find vector $\overrightarrow{\mathrm{AB}}$ ***(M1)***

$\overrightarrow{\mathrm{AB}} = \begin{pmatrix}-1\\-2\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\1\end{pmatrix}$ $(\mu\in\mathbb{R})$ ***A1***

$\left|\overrightarrow{\mathrm{AB}}\right| = \sqrt{(-1+\mu)^2+(-2-\mu)^2+(5+\mu)^2}$ $\left(=\sqrt{3\mu^2+12\mu+30}\right)$ ***M1***

EITHER

$\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\left|\overrightarrow{\mathrm{AB}}\right|^2\right)=0\Rightarrow 6\mu+12=0\Rightarrow \mu=-2$ ***A1***

OR

$\left|\overrightarrow{\mathrm{AB}}\right| = \sqrt{3(\mu+2)^2+18}$ to obtain $\mu=-2$ ***A1***

THEN

minimum distance is $\sqrt{18}\left(=3\sqrt{2}\right)$ ***A1***

[5 marks]

continued…

Question 8 continued

METHOD 3

let A be $(3,2,-1)$ on l_1 and let B be $(2+\mu,-\mu,4+\mu)$ on l_2 ***(M1)***

(or let A be $(2,0,4)$ on l_2 and let B be $(3+2\lambda,2-2\lambda,-1+2\lambda)$ on l_1)

$\overrightarrow{\mathrm{AB}}=\begin{pmatrix}-1\\-2\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\1\end{pmatrix}$ $(\mu\in\mathbb{R})$ (or $\overrightarrow{\mathrm{AB}}=\begin{pmatrix}2\lambda+1\\-2\lambda+2\\2\lambda-5\end{pmatrix}$) ***A1***

$\begin{pmatrix}\mu-1\\-\mu-2\\\mu+5\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$ (or $\begin{pmatrix}2\lambda+1\\-2\lambda+2\\2\lambda-5\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$) ***M1***

$\mu=-2$ or $\lambda=1$ ***A1***

minimum distance is $\sqrt{18}\left(=3\sqrt{2}\right)$ ***A1***

[5 marks]
Total [8 marks]

9. $u = \sin x \Rightarrow \mathrm{d}u = \cos x \, \mathrm{d}x$ (or equivalent) ***A1***

$= \int \frac{u}{u^2 - u - 2} \mathrm{d}u$ ***A1***

attempt to use partial fractions ***M1***

$$\left(\frac{u}{(u+1)(u-2)} \equiv \frac{A}{u+1} + \frac{B}{u-2} \Rightarrow u \equiv A(u-2) + B(u+1) \right)$$

Valid attempt to solve for A and B ***(M1)***

$A = \frac{1}{3}$ and $B = \frac{2}{3}$ ***A1***

$$\frac{u}{(u+1)(u-2)} \equiv \frac{1}{3(u+1)} + \frac{2}{3(u-2)}$$

$\int \left(\frac{1}{3(u+1)} + \frac{2}{3(u-2)} \right) \mathrm{d}u = \frac{1}{3} \ln|u+1| + \frac{2}{3} \ln|u-2| (+C)$ (or equivalent) ***A1***

Note: Condone the absence of $+C$ or lack of moduli here but not in the final answer.

$= \frac{1}{3} \ln|\sin x + 1| + \frac{2}{3} \ln|\sin x - 2| + C$ ***A1***

Note: Condone further simplification of the correct answer.

[7 marks]

Section B

10. (a) $6+6\cos x=0$ (or setting their $f'(x)=0$) ***(M1)***

$\cos x=-1$ (or $\sin x=0$)

$x=\pi,\ x=3\pi$ ***A1A1***

[3 marks]

(b) attempt to integrate $\int_{\pi}^{3\pi}(6+6\cos x)\mathrm{d}x$ ***(M1)***

$=[6x+6\sin x]_{\pi}^{3\pi}$ ***A1A1***

substitute their limits into their integrated expression and subtract ***(M1)***

$=(18\pi+6\sin 3\pi)-(6\pi+6\sin\pi)$

$=(6(3\pi)+0)-(6\pi+0)\ \ (=18\pi-6\pi)$ ***A1***

area $=12\pi$ ***AG***

[5 marks]

(c) attempt to substitute into the formula for surface area (including base) ***(M1)***

$\pi(2^2)+\pi(2)(l)=12\pi$ ***(A1)***

$4\pi+2\pi l=12\pi$

$2\pi l=8\pi$

$l=4$ ***A1***

[3 marks]

continued…

Question 10 continued

(d) valid attempt to find the height of the cone ***(M1)***

e.g. $2^2 + h^2 = (\text{their } l)^2$

$h = \sqrt{12} \ \left(= 2\sqrt{3}\right)$ ***(A1)***

attempt to use $V = \frac{1}{3}\pi r^2 h$ with their values substituted ***M1***

$\left(\frac{1}{3}\pi\left(2^2\right)\left(\sqrt{12}\right)\right)$

$\text{volume} = \frac{4\pi\sqrt{12}}{3}\left(= \frac{8\pi\sqrt{3}}{3} = \frac{8\pi}{\sqrt{3}}\right)$ ***A1***

[4 marks]
Total [15 marks]

11. (a) $\frac{\mathrm{d}v}{\mathrm{d}t} = -(1+v)$ ***(A1)***

$\int 1\mathrm{d}t = \int -\frac{1}{1+v}\,\mathrm{d}v$ (or equivalent / use of integrating factor) ***M1***

$t = -\ln(1+v)(+C)$ ***A1***

EITHER

attempt to find C with initial conditions $t = 0, v = v_0$ ***M1***

$C = \ln(1+v_0)$

$t = \ln(1+v_0) - \ln(1+v)$

$t = \ln\left(\frac{1+v_0}{1+v}\right) \Rightarrow \mathrm{e}^t = \frac{1+v_0}{1+v}$ ***A1***

$\mathrm{e}^t(1+v) = 1+v_0$

$1+v = (1+v_0)\mathrm{e}^{-t}$ ***A1***

$v(t) = (1+v_0)\mathrm{e}^{-t} - 1$ ***AG***

continued…

OR

$$t - C = -\ln\left(1+v\right) \Rightarrow \mathrm{e}^{t-C} = \frac{1}{\left(1+v\right)}$$

Attempt to find C with initial conditions $t = 0, v = v_0$ ***M1***

$$\mathrm{e}^{-C} = \frac{1}{\left(1+v_0\right)} \Rightarrow C = \ln\left(1+v_0\right)$$

$$t - \ln(1+v_0) = -\ln\left(1+v\right) \Rightarrow t = \ln(1+v_0) - \ln\left(1+v\right)$$

$$t = \ln\left(\frac{1+v_0}{1+v}\right) \Rightarrow \mathrm{e}^{t} = \frac{1+v_0}{1+v}$$ ***A1***

$$\mathrm{e}^{t}\left(1+v\right) = 1+v_0$$

$$1+v = \left(1+v_0\right)\mathrm{e}^{-t}$$ ***A1***

$$v\left(t\right) = \left(1+v_0\right)\mathrm{e}^{-t} - 1$$ ***AG***

OR

$$t - C = -\ln\left(1+v\right) \Rightarrow \mathrm{e}^{-t+C} = 1+v$$ ***A1***

$$k\mathrm{e}^{-t} - 1 = v$$

Attempt to find k with initial conditions $t = 0, v = v_0$ ***M1***

$$k = 1+v_0$$

$$\mathrm{e}^{-t}\left(1+v_0\right) = 1+v$$ ***A1***

$$v\left(t\right) = \left(1+v_0\right)\mathrm{e}^{-t} - 1$$ ***AG***

Note: condone use of modulus within the ln function(s)

[6 marks]

continued…

Question 11 continued

(b) (i) recognition that when $t = T, v = 0$ ***M1***

$$(1+v_0)\mathrm{e}^{-T} - 1 = 0 \Rightarrow \mathrm{e}^{-T} = \frac{1}{1+v_0}$$ ***A1***

$$\mathrm{e}^{T} = 1 + v_0$$ ***AG***

Note: Award ***M1A0*** for substituting $v_0 = \mathrm{e}^T - 1$ into v and showing that $v = 0$.

(ii) $s(t) = \int v(t)\ \mathrm{d}t\ \left(= \int \left((1+v_0)\mathrm{e}^{-t} - 1\right)\mathrm{d}t\right)$ ***(M1)***

$= -(1+v_0)\mathrm{e}^{-t} - t(+D)$ ***A1***

($t = 0, s = 0$ so) $D = 1 + v_0$ ***A1***

$s(t) = -(1+v_0)\mathrm{e}^{-t} - t + 1 + v_0$

at $s_{\max}$, $\mathrm{e}^{T} = 1 + v_0 \Rightarrow T = \ln(1+v_0)$

Substituting into $s(t)\left(= -(1+v_0)\mathrm{e}^{-t} - t + 1 + v_0\right)$ ***M1***

$$s_{\max} = -(1+v_0)\left(\frac{1}{1+v_0}\right) - \ln(1+v_0) + v_0 + 1$$ ***A1***

$$\left(s_{\max} = v_0 - \ln(1+v_0)\right)$$

[7 marks]

(c) **METHOD 1**

$v(T-k) = (1+v_0)\mathrm{e}^{-T}\mathrm{e}^{k} - 1$ ***(M1)***

$= (1+v_0)\left(\frac{1}{1+v_0}\right)\mathrm{e}^{k} - 1$ ***A1***

$= \mathrm{e}^{k} - 1$ ***AG***

[2 marks]

METHOD 2

$v(T-k) = (1+v_0)\mathrm{e}^{-(T-k)} - 1$

$= e^{T}\mathrm{e}^{-(T-k)} - 1$ ***M1***

$= e^{T-T+k} - 1$ ***A1***

$= \mathrm{e}^{k} - 1$ ***AG***

[2 marks]

continued…

Question 11 continued

(d) **METHOD 1**

$v(T+k)=(1+v_0)\mathrm{e}^{-T}\mathrm{e}^{-k}-1$ ***(A1)***

$=\mathrm{e}^{-k}-1$ ***A1***

[2 marks]

METHOD 2

$v(T+k)=(1+v_0)\mathrm{e}^{-(T+k)}-1$ ***(A1)***

$=e^{T}\mathrm{e}^{-(T+k)}-1$

$=e^{T-T-k}-1$

$=\mathrm{e}^{-k}-1$ ***A1***

[2 marks]

(e) **METHOD 1**

$v(T-k)+v(T+k)=\mathrm{e}^{k}+\mathrm{e}^{-k}-2$ ***A1***

attempt to express as a square ***M1***

$=\left(\mathrm{e}^{\frac{k}{2}}-\mathrm{e}^{-\frac{k}{2}}\right)^{2}(\geq 0)$ ***A1***

so $v(T-k)+v(T+k)\geq 0$ ***AG***

[3 marks]

METHOD 2

$v(T-k)+v(T+k)=\mathrm{e}^{k}+\mathrm{e}^{-k}-2$ ***A1***

Attempt to solve $\frac{\mathrm{d}}{\mathrm{d}k}\left(\mathrm{e}^{k}+\mathrm{e}^{-k}\right)=0$ $(\Rightarrow k=0)$ ***M1***

minimum value of 2, (when $k=0$), hence $\mathrm{e}^{k}+\mathrm{e}^{-k}\geq 2$ ***R1***

so $v(T-k)+v(T+k)\geq 0$ ***AG***

[3 marks]
Total [20 marks]

12. (a) **EITHER**

horizontal stretch/scaling with scale factor $\frac{1}{2}$

Note: Do not allow 'shrink' or 'compression'

followed by a horizontal translation/shift $\frac{1}{2}$ units to the left ***A2***

Note: Do not allow 'move'

OR

horizontal translation/shift 1 unit to the left

followed by horizontal stretch/scaling with scale factor $\frac{1}{2}$ ***A2***

THEN

vertical translation/shift up by $\frac{\pi}{4}$ (or translation through $\begin{pmatrix} 0 \\ \frac{\pi}{4} \end{pmatrix}$) ***A1***

(may be seen anywhere)

[3 marks]

(b) let $\alpha = \arctan p$ and $\beta = \arctan q$ ***M1***

$p = \tan\alpha$ and $q = \tan\beta$ ***(A1)***

$\tan(\alpha+\beta) = \frac{p+q}{1-pq}$ ***A1***

$\alpha+\beta = \arctan\left(\frac{p+q}{1-pq}\right)$ ***A1***

so $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where $p, q > 0$ and $pq < 1$ ***AG***

[4 marks]

continued…

Question 12 continued

(c) **METHOD 1**

$\frac{\pi}{4} = \arctan 1$ (or equivalent) ***A1***

$\arctan\left(\frac{x}{x+1}\right) + \arctan 1 = \arctan\left(\frac{\frac{x}{x+1}+1}{1-\frac{x}{x+1}(1)}\right)$ ***A1***

$= \arctan\left(\frac{\frac{x+x+1}{x+1}}{\frac{x+1-x}{x+1}}\right)$ ***A1***

$= \arctan(2x+1)$ ***AG***

[3 marks]

METHOD 2

$\tan\frac{\pi}{4} = 1$ (or equivalent) ***A1***

Consider $\arctan(2x+1) - \arctan\left(\frac{x}{x+1}\right) = \frac{\pi}{4}$

$\tan\left(\arctan(2x+1) - \arctan\left(\frac{x}{x+1}\right)\right)$

$= \arctan\left(\frac{2x+1-\frac{x}{x+1}}{1+\frac{x(2x+1)}{x+1}}\right)$ ***A1***

$= \arctan\left(\frac{(2x+1)(x+1)-x}{x+1+x(2x+1)}\right)$ ***A1***

$= \arctan 1$ ***AG***

[3 marks]

continued…

Question 12 continued

METHOD 3

$$\tan\left(\arctan\left(2x+1\right)\right)=\tan\left(\arctan\left(\frac{x}{x+1}\right)+\frac{\pi}{4}\right)$$

$\tan\frac{\pi}{4}=1$ (or equivalent) ***A1***

LHS = $2x+1$ ***A1***

RHS = $\dfrac{\frac{x}{x+1}+1}{1-\frac{x}{x+1}}\left(=2x+1\right)$ ***A1***

[3 marks]

Question 12 continued

(d) let $\mathrm{P}(n)$ be the proposition that $\sum_{r=1}^{n}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{n}{n+1}\right)$ for $n\in\mathbb{Z}^+$

consider $\mathrm{P}(1)$:

when $n=1$, $\sum_{r=1}^{1}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{1}{2}\right)=\mathrm{RHS}$ and so $\mathrm{P}(1)$ is true **R1**

assume $\mathrm{P}(k)$ is true, ie. $\sum_{r=1}^{k}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{k}{k+1}\right)$ $\left(k\in\mathbb{Z}^+\right)$ ***M1***

Note: Award ***M0*** for statements such as "let $n=k$".
Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

consider $\mathrm{P}(k+1)$:

$$\sum_{r=1}^{k+1}\arctan\left(\frac{1}{2r^2}\right)=\sum_{r=1}^{k}\arctan\left(\frac{1}{2r^2}\right)+\arctan\left(\frac{1}{2(k+1)^2}\right)$$ **(*M1*)**

$$=\arctan\left(\frac{k}{k+1}\right)+\arctan\left(\frac{1}{2(k+1)^2}\right)$$ ***A1***

$$=\arctan\left(\frac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\left(\frac{k}{k+1}\right)\left(\frac{1}{2(k+1)^2}\right)}\right)$$ ***M1***

$$=\arctan\left(\frac{(k+1)(2k^2+2k+1)}{2(k+1)^3-k}\right)$$ ***A1***

Note: Award ***A1*** for correct numerator, with $(k+1)$ factored. Denominator does not need to be simplified

$$=\arctan\left(\frac{(k+1)(2k^2+2k+1)}{2k^3+6k^2+5k+2}\right)$$ ***A1***

Note: Award ***A1*** for denominator correctly expanded. Numerator does not need to be simplified. These two ***A*** marks may be awarded in any order

$$=\arctan\left(\frac{(k+1)(2k^2+2k+1)}{(k+2)(2k^2+2k+1)}\right)=\arctan\left(\frac{k+1}{k+2}\right)$$ ***A1***

Note: The word 'arctan' must be present to be able to award the last three A marks

continued…

Question 12 continued

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so
$\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ ***R1***

Note:	Award the final ***R1*** mark provided at least four of the previous marks have been awarded.
Note:	To award the final ***R1***, the truth of $\mathrm{P}(k)$ must be mentioned. '$\mathrm{P}(k)$ implies $\mathrm{P}(k+1)$' is insufficient to award the mark.

[9 marks]
Total [19 marks]