IBO

INTERNATIONAL BACCALAUREATE ORGANIZATION

DIPLOMA PROGRAMME

Mathematics higher level

For first examinations in 2001

Mathematics Higher Level
February 1998

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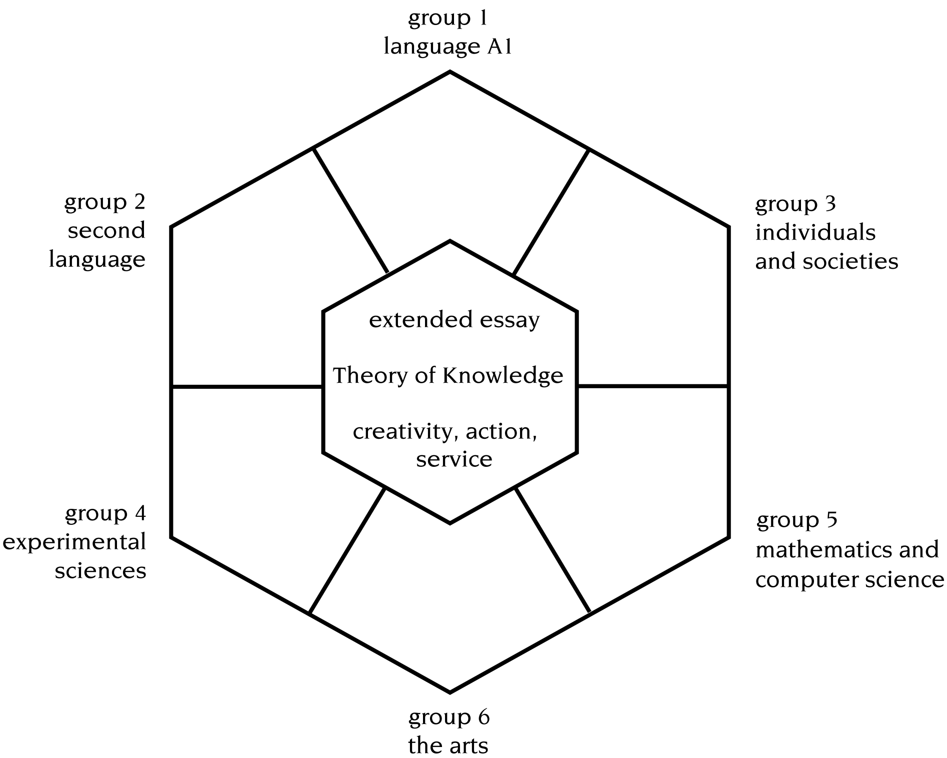
INTRODUCTION

The International Baccalaureate Diploma Programme is a rigorous pre-university course of studies, leading to examinations, that meets the needs of highly motivated secondary school students between the ages of 16 and 19 years. Designed as a comprehensive two-year curriculum that allows its graduates to fulfil requirements of various national education systems, the diploma model is based on the pattern of no single country but incorporates the best elements of many. The programme is available in English, French and Spanish.

The curriculum is displayed in the shape of a hexagon with six academic areas surrounding the core. Subjects are studied concurrently and students are exposed to the two great traditions of learning: the humanities and the sciences.



Diploma Programme candidates are required to select one subject from each of the six subject groups. At least three and not more than four are taken at higher level (HL), the others at standard level (SL). Higher level courses represent 240 teaching hours; SL courses cover 150 hours. By arranging work in this fashion, students are able to explore some subjects in depth and some more broadly over the two-year period; this is a deliberate compromise between the early specialisation preferred in some national systems and the breadth found in others.

Distribution requirements ensure that the science-orientated student is challenged to learn a foreign language and that the natural linguist becomes familiar with science laboratory procedures. While overall balance is maintained, flexibility in choosing higher level concentrations allows the student to pursue areas of personal interest and to meet special requirements for university entrance.

Successful Diploma Programme candidates meet three requirements in addition to the six subjects. The interdisciplinary Theory of Knowledge (TOK) course is designed to develop a coherent approach to learning which transcends and unifies the academic areas and encourages appreciation of other cultural perspectives. The extended essay of some 4000 words offers the opportunity to investigate a topic of special interest and acquaints students with the independent research and writing skills expected at university. Participation in the Creativity, Action, Service (CAS) requirement encourages students to be involved in sports, artistic pursuits and community service work.

For first examinations in 2001

NATURE OF THE SUBJECT

Introduction

The nature of mathematics can be summarized in a number of ways; for example, as a well-defined body of knowledge, as an abstract system of ideas or as a useful tool. For many people it is probably a combination of these, but there is no doubt that mathematical knowledge provides an important key to understanding the world in which we live. Mathematics can enter our lives in a number of ways: buying produce in the market, consulting a timetable, reading a newspaper, timing a process or estimating a length. For most people mathematics also extends into their chosen profession: artists need to learn about perspective; musicians need to appreciate the mathematical relationships within and between different rhythms; economists need to recognize trends in financial dealings; and engineers need to take account of stress patterns. Scientists view mathematics as a language that is vital to our understanding of events that occur in the natural world. Other people are challenged by the logical methods of mathematics and the adventure in reason that mathematical proof has to offer. Still others appreciate mathematics as an aesthetic experience or even as a cornerstone of philosophy. The prevalence of mathematics in people's lives thus provides a clear and sufficient rationale for making the study of this subject compulsory within the IB diploma.

Since individual students have different needs, interests and abilities, the International Baccalaureate Organization (IBO) offers a number of different courses in mathematics. These are targeted at students who wish to study mathematics in depth, either as a subject in its own right or in order to pursue their interests in areas related to mathematics, those who wish to gain a degree of understanding and competence in order to understand better their approach to other subjects, and those who may not be aware that mathematics has relevance in their studies and in their future lives. Each course is designed to meet the needs of a particular group of students and therefore great care should be exercised in selecting the one which is most appropriate for an individual student.

In making the selection, individual students should be advised to take account of the following considerations.

- Their own abilities in mathematics and the type of mathematics in which they can be successful.

- Their own interest in mathematics with respect to the areas which hold an appeal.

- Their other choices of subjects within the framework of the Diploma Programme.

- Their future academic plans in terms of the subjects they wish to study.

- Their choice of career.

Teachers are expected to assist with the selection process and to offer advice to students on choosing the most appropriate subject from group 5.

Mathematics higher level

Mathematics, available as a higher level (HL) subject only, caters for students with a good background in mathematics who are competent in a range of analytical and technical skills. The majority of these students will be expecting to include mathematics as a major component of their university studies, either as a subject in its own right or within courses such as physics, engineering and technology. Others may take this subject because they have a strong interest in mathematics and enjoy meeting its challenges and engaging with its problems.

The nature of the subject is such that it focuses on developing important mathematical concepts in a comprehensible and coherent way. This is achieved by means of a carefully balanced approach: students are encouraged to apply their mathematical knowledge to solving problems set in a variety of meaningful contexts while, at the same time, being introduced to important concepts of rigour and proof.

Students embarking on this course should expect to develop insight into mathematical form and structure in their studies, and should be intellectually equipped to appreciate the links between parallel structures in different topic areas. They should also be encouraged to develop the skills needed to continue their mathematical growth in other learning environments.

The internally assessed component, the portfolio, offers students a framework for developing independence in their mathematical development through engaging in the following activities: mathematical investigation, extended closed-problem solving and mathematical modelling. Students will thus be provided with the means to ask their own questions about mathematics and be given the chance to explore different ways of arriving at a solution, either through experimenting with the techniques at their disposal or by researching new methods. This process also allows students to work without the time constraints of a written examination and to acquire ownership of a part of the course.

This course is clearly a demanding one, requiring students to study a broad range of mathematical topics through a number of different approaches and to varying degrees of depth. Students wishing to study mathematics in a less rigorous environment should therefore opt for one of the standard level courses: mathematical methods or mathematical studies.

AIMS

The aims of all courses in group 5 are to enable candidates to:

- appreciate the international dimensions of mathematics and the multiplicity of its cultural and historical perspectives
- foster enjoyment from engaging in mathematical pursuits, and to develop an appreciation of the beauty, power and usefulness of mathematics
- develop logical, critical and creative thinking in mathematics
- develop mathematical knowledge, concepts and principles
- employ and refine the powers of abstraction and generalization
- develop patience and persistence in problem-solving
- have an enhanced awareness of, and utilize the potential of, technological developments in a variety of mathematical contexts
- communicate mathematically, both clearly and confidently, in a variety of contexts.

OBJECTIVES

Having followed any one of the courses in group 5, candidates will be expected to:

- know and use mathematical concepts and principles
- read and interpret a given problem in appropriate mathematical terms
- organize and present information/data in tabular, graphical and/or diagrammatic forms
- know and use appropriate notation and terminology
- formulate a mathematical argument and communicate it clearly
- select and use appropriate mathematical techniques
- understand the significance and reasonableness of results
- recognize patterns and structures in a variety of situations and draw inductive generalizations
- demonstrate an understanding of, and competence in, the practical applications of mathematics
- use appropriate technological devices as mathematical tools.

SYLLABUS OUTLINE

The mathematics higher level (HL) syllabus consists of the study of eight **core** topics and one **option**.

Part I: Core — 195 hours

All topics in the core are compulsory. Candidates are required to study all the sub-topics in each of the eight topics in this part of the syllabus as listed in the Syllabus Details.

1	Number and algebra	20 hours
2	Functions and equations	25 hours
3	Circular functions and trigonometry	25 hours
4	Vector geometry	25 hours
5	Matrices and transformations	20 hours
6	Statistics	10 hours
7	Probability	20 hours
8	Calculus	50 hours

Part II: Options — 35 hours

Candidates are required to study all the sub-topics in ***one*** *of the following options as listed in the Syllabus Details.*

9	Statistics	35 hours
10	Sets, relations and groups	35 hours
11	Discrete mathematics	35 hours
12	Analysis and approximation	35 hours
13	Euclidean geometry and conic sections	35 hours

Portfolio **10 hours**

Three assignments, based on different areas of the syllabus, representing each of the following activities:

- mathematical investigation
- extended closed-problem solving
- mathematical modelling

SYLLABUS DETAILS

Format of the syllabus

The syllabus is formatted into three columns labelled Content, Amplifications/Exclusions and Teaching Notes.

- **Content:** the first column lists, under each topic, the sub-topics to be covered.

- **Amplifications/Exclusions:** the second column contains more explicit information on specific sub-topics listed in the first column. This helps to define what is required and what is not required in terms of preparing for the examination.

- **Teaching Notes:** the third column provides useful suggestions for teachers. It is not mandatory that these suggestions be followed.

Course of study

Teachers are required to teach all the sub-topics listed under the eight topics in the core, together with all the sub-topics in the chosen option.

It is not necessary, nor desirable, to teach the topics in the core in the order in which they appear in the Syllabus Outline and Syllabus Details. Neither is it necessary to teach all the topics in the core before starting to teach an option. Teachers are therefore strongly advised to draw up a course of study, tailored to the needs of their students, which integrates the areas covered by both the core and the chosen option.

Integration of portfolio assignments

The three assignments for the portfolio, based on the three activities (mathematical investigation, extended closed-problem solving and mathematical modelling), should be incorporated into the course of study, and should relate directly to topics in the syllabus. Full details are given in Assessment Details, Portfolio.

Time allocation

The recommended teaching time for a higher level subject is 240 hours. For mathematics HL, it is expected that 10 hours will be spent on work for the portfolio. The time allocations given in the Syllabus Outline and Syllabus Details are approximate, and are intended to suggest how the remaining 230 hours allowed for teaching the syllabus might be allocated. However, the exact time spent on each topic will depend on a number of factors, including the background knowledge and level of preparedness of each student. Teachers should therefore adjust these timings to correspond with the needs of their students.

Use of calculators

Candidates are required to have access to a graphic display calculator at all times during the course, both inside and out of the classroom. Regulations concerning the types of calculators allowed are provided in the *Vade Mecum*.

Formulae booklet and statistical tables (third edition, February 2001)

As each candidate is required to have access to clean copies of the IBO formulae booklet and statistical tables during the examination, it is recommended that teachers ensure candidates are familiar with the contents of these documents from the beginning of the course. The booklet and tables are provided by IBCA and are published separately.

Resource list

A resource list is available for mathematics HL on the online curriculum centre. This list provides details of, for example, texts, software packages and videos which are considered by teachers to be appropriate for use with this course. It will be updated on a regular basis.

Teachers can at any time add any materials to this list which they consider to be appropriate for candidate use or as reference material for teachers.

1 Core: number and algebra

Teaching time: 20 hours

The aims of this section are to introduce important results and methods of proof in algebra, and to extend the concept of number to include complex numbers.

Content	Amplifications/Exclusions	Teaching Notes
1.1 Arithmetic sequences and series; sum of finite arithmetic series; geometric sequences and series; sum of finite and infinite geometric series. Applications of the above.	Included: sigma notation, ie $\sum_{i=1}^{n} a_i$. Included: applications of sequences and series to compound interest and population growth.	Generation of terms and partial sums by iterating on a calculator can be useful. Link with limits and convergence in §8.1.
1.2 Exponents and logarithms: laws of exponents; laws of logarithms.	Included: change of base, ie $\log_b a = \frac{\log_c a}{\log_c b}$.	This topic is developed further in §2.9.
1.3 The binomial theorem: expansion of $(a+b)^n,\ n \in \mathbf{N}$.	Included: the formulae $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.	Although only the notation $\binom{n}{r}$ will be used in examination papers, candidates will need to be aware of alternative notation used in textbooks and on calculators, eg ${}_n\mathrm{C}_r$, $n\mathrm{C}r$, C_r^n . Link with mathematical induction in §1.4. Link with De Moivre's theorem in §1.7. Link with counting principles in §7.5. Link with binomial distribution in §7.7. Link with limits and convergence in §8.1.
1.4 Proof by mathematical induction. Forming conjectures to be proved by mathematical induction.	Included: proofs of standard results for sums of squares and cubes of natural numbers.	Link with binomial theorem in §1.3 and De Moivre's theorem in §1.7.
1.5 Complex numbers: the number $\mathrm{i} = \sqrt{-1}$; the terms real part, imaginary part, conjugate, modulus and argument; the forms $z = a + \mathrm{i}b$ and $z = r(\cos\theta + \mathrm{i}\sin\theta)$. The complex plane.	Included: cartesian and polar forms of a complex number.	

1 Core: number and algebra (continued)

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
1.6 Sums, products and quotients of complex numbers.	Included: multiplication by i as a rotation of 90° in the complex plane.	Link with transformation of vectors in §5.5.
1.7 De Moivre's theorem (proof by mathematical induction). Powers and roots of a complex number.		Link with binomial theorem in §1.3. Link with proof by induction in §1.4.
1.8 Conjugate roots of polynomial equations with real coefficients.	Not required: equations with complex coefficients.	

2 Core: functions and equations

Teaching time: 25 hours

The aims of this section are to introduce methods of solution for different types of equations, to explore the notion of function as a unifying theme in mathematics, to study certain functions in more depth and to explore the transformations of the graphical representations of functions.

Content	Amplifications/Exclusions	Teaching Notes
2.1 Concept of function $f: x \mapsto f(x)$: domain, range; image (value). Composite functions $f \circ g$; identity function; inverse function f^{-1}. Domain restriction. The graph of a function; its equation $y = f(x)$.	In examinations: if the domain is the set of real numbers then the statement "$x \in \mathbf{R}$" will be omitted. Included: formal definition of a function; the terms "one-one", and "many-one". Not required: the term "codomain".	General examples: for $x \mapsto \sqrt{2-x}$, domain is $x \le 2$, range is $y \ge 0$; for $x \mapsto$ "distance from nearest integer", domain is $\mathbf{R}$, range is $0 \le y \le 0.5$. Example of domain restriction: $x \mapsto \sqrt{x-3}$ is the inverse of $x \mapsto x^2 + 3, x \ge 0$, but $x \mapsto -\sqrt{x-3}$ is the inverse of $x \mapsto x^2 + 3, x < 0$. Note that the composite function $(f \circ g)(x)$ is defined as $f(g(x))$. Link with the chain rule for composite functions in §8.3.
2.2 Function graphing skills: use of a graphic display calculator to graph a variety of functions. Appropriate choice of "window", use of "zoom" and "trace" (or equivalent) to locate points to a given accuracy; use of "connected" and "dot" (or equivalent) modes as appropriate. Solution of $f(x) = 0$ to a given accuracy.	Included: identification of horizontal and vertical asymptotes; use of the calculator to find maximum and minimum points. On examination papers: questions may be set which require the graphing of functions which do not explicitly appear on the syllabus.	Calculator settings should be chosen appropriately to avoid, for example, interpolation across a vertical asymptote. These graphing skills should be utilized throughout the syllabus as appropriate. Link with maximum and minimum problems in §8.6.

2 Core: functions and equations (continued)

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
2.3 Transformations of graphs: translations; stretches; reflections in the axes. The graph of f^{-1} as the reflection in the line $y = x$ of the graph of f. Absolute value function $\lvert f \rvert$. The graph of $\dfrac{1}{f(x)}$ from $f(x)$.	Translations: $y = f(x) + b$; $y = f(x - a)$. Stretches: $y = pf(x)$; $y = f\left(x/q\right)$. Reflections (in the x-axis and y-axis): $y = f(-x)$; $y = -f(x)$. Included: $y = \lvert f(x) \rvert$, $y = f(\lvert x \rvert)$.	Examples: $y = x^2$ may be used to obtain $y = (x-3)^2 + 5$ by a translation of $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$, or $y = \sin x$ may be used to obtain $y = 3\sin\left(x/2\right)$ by a two-way stretch. Link with quadratic functions in §2.5. Link with exponential functions in §2.9. Link with circular functions in §3.3. Link with matrix transformations in §5.5.
2.4 The reciprocal function $x \mapsto \dfrac{1}{x}, x \neq 0$: its graph; its self-inverse nature.		
2.5 The quadratic function $x \mapsto ax^2 + bx + c$: its graph. The form $x \mapsto a(x-h)^2 + k$: vertex (h, k) and y-intercept $(0, c)$. The form $x \mapsto a(x-p)(x-q)$: x-intercepts $(p, 0)$ and $(q, 0)$.	Included: rational coefficients only.	Link the second form, "completing the square", with transformations of functions in §2.3, ie $y = a(x-h)^2 + k$ as $y = x^2$ transformed.
2.6 Solution of $f(x) = g(x)$, f, g linear or quadratic.	In examinations: questions demanding elaborate factorization techniques will not be set. Included: knowledge of the significance of the discriminant $\Delta = b^2 - 4ac$ for the solution set in the three cases $\Delta > 0, \Delta = 0, \Delta < 0$.	

2 Core: functions and equations (continued)

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
2.7 Inequalities in one variable, including their graphical representation. Solution of $f(x) \geq g(x)$, f, g linear or quadratic.	Included: cases where cross multiplication is not appropriate, eg $\frac{2}{x-2} > \frac{1}{x-3}$; the use of the absolute value sign in inequalities.	
2.8 Polynomial functions. The factor and remainder theorems, with application to the solution of polynomial equations and inequations.	Included: the significance of multiple roots.	The use of synthetic division may be encouraged in finding zeros, remainders and values. The use of graphic display calculators should be encouraged in finding zeros through trace or calculation.
2.9 The exponential function $x \mapsto a^x, a > 0$: its domain and range. The inverse function $x \mapsto \log_a x$. Graphs of $y = a^x$ and $y = \log_a x$. Solution of $a^x = b$.	Included: for the domain of a^x only rational x need be considered. Included: knowledge that $\log_a a^x = x = a^{\log_a x}$. Included: knowledge that $a^x = b \Leftrightarrow x = \log_a b$.	Link with the laws of exponents and logarithms in §1.2. Note that the graph of $y = a^x$ reflected in the line $y = x$ gives the graph of $y = \log_a x$; link with transformations of graphs in §2.3. This topic may be linked with the applications of geometric sequences in §1.1.
2.10 The functions $x \mapsto e^x, x \mapsto \ln x$. Application to the solution of equations based on problems of growth and decay.	Included: a^x expressed as $e^{x \ln a}$. Included: applications to population growth and compound interest (eg doubling-times), and radioactive decay (eg half-life).	Link with differential equations in §8.11.

3 Core: circular functions and trigonometry

Teaching time: 25 hours

The aims of this section are to use trigonometry to solve general triangles, to explore the behaviour of circular functions both graphically and algebraically and to introduce some important identities in trigonometry.

Content	Amplifications/Exclusions	Teaching Notes
3.1 The circle: radian measure of angles; length of an arc; area of a sector.	Included: radian measure expressed as multiples of π.	Note that $2\pi r$ generalizes to θr, πr^2 generalizes to $\frac{1}{2}\theta r^2$.
3.2 Definition of $(\cos\theta,\ \sin\theta)$ in terms of the unit circle. The Pythagorean identities: $\cos^2\theta+\sin^2\theta=1$; $1+\tan^2\theta=\sec^2\theta$; $1+\cot^2\theta=\csc^2\theta$.	Included: given $\sin\theta$, finding possible values of $\cos\theta$.	
3.3 The six circular functions: $x\mapsto\sin x$, $x\mapsto\cos x$, $x\mapsto\tan x$, $x\mapsto\csc x$, $x\mapsto\sec x$, $x\mapsto\cot x$; their domains and ranges; their periodic nature, and their graphs. The inverse functions $x\mapsto\arcsin x$, $x\mapsto\arccos x$, $x\mapsto\arctan x$; their domains and ranges, and their graphs.	In examinations: radian measure should be assumed unless otherwise indicated (eg $x\mapsto\sin x^\circ$).	Although only the notations $\arcsin x$, etc will be used on examination papers, candidates will need to be aware of alternative notations used on calculators. The graph of $y=a\sin b(x+c)$ may be presented as a transformation of $y=\sin x$. Link with inverse functions in §2.3.
3.4 Addition, double-angle and half-angle formulae: $\sin(A+B)$, etc; $\sin 2A$, etc; $\sin\frac{1}{2}A$, etc. The compound formula $a\cos x\pm b\sin x=R\cos(x\mp\alpha)$.	Included: proof of addition and double-angle formulae. Not required: formal proof of the compound formula.	

3 Core: circular functions and trigonometry (continued)

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
3.5 Composite functions of the form $f(x) = a\sin b(x+c)$; solutions of $f(x) = k$ in a given finite interval. Solution of equations leading to quadratic or linear equations in $\sin x$, etc. Graphical interpretation of the above.		
3.6 Solution of triangles. The cosine rule: $c^2 = a^2 + b^2 - 2ab\cos \mathrm{C}$. The sine rule: $\dfrac{a}{\sin \mathrm{A}} = \dfrac{b}{\sin \mathrm{B}} = \dfrac{c}{\sin \mathrm{C}}$. Area of a triangle as $\dfrac{1}{2}ab\sin \mathrm{C}$, etc.	Included: the derivation of the sine rule from the formula for the area of the triangle; the ambiguous case of the sine rule; applications to practical problems in two dimensions and three dimensions.	Appreciation of Pythagoras' theorem as a special case of the cosine rule. Link with the cosine rule in scalar product form in §4.3.

4 Core: vector geometry

Teaching time: 25 hours

The aims of this section are to introduce the use of vectors in two and three dimensions, to facilitate solution of problems involving points, lines and planes, and to enable the associated angles, distances and areas to be calculated.

Content	Amplifications/Exclusions	Teaching Notes
4.1 Vectors as displacements in the plane and in three dimensions, $\boldsymbol{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$. Components of a vector; column representation. The sum of two vectors; the zero vector; the inverse vector, $-\boldsymbol{v}$. Multiplication by a scalar, $k\boldsymbol{v}$. Magnitude of a vector, $\lvert\boldsymbol{v}\rvert$. Position vectors $\overrightarrow{\mathrm{OA}}=\boldsymbol{a}$. Unit vectors including $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$.	Note: components are with respect to the standard basis $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$: $\boldsymbol{v}=v_1\boldsymbol{i}+v_2\boldsymbol{j}+v_3\boldsymbol{k}$. Included: the difference of $\boldsymbol{v}$ and $\boldsymbol{w}$ as $\boldsymbol{v}-\boldsymbol{w}=\boldsymbol{v}+(-\boldsymbol{w})$. Included: the vector $\overrightarrow{\mathrm{AB}}$ expressed as $\overrightarrow{\mathrm{OB}}-\overrightarrow{\mathrm{OA}}=\boldsymbol{b}-\boldsymbol{a}$.	Vector sums and differences can be represented by the diagonals of a parallelogram. Multiplication by a scalar can be illustrated by enlarging the vector parallelogram. Applications to simple geometric figures, eg ABCD is a quadrilateral and $\overrightarrow{\mathrm{AB}}=-\overrightarrow{\mathrm{CD}}\Rightarrow \mathrm{ABCD}$ is a parallelogram.
4.2 The scalar product of two vectors $\boldsymbol{u}\cdot\boldsymbol{v}=u_1v_1+u_2v_2+u_3v_3$. Properties of the scalar product $\boldsymbol{v}\cdot\boldsymbol{w}=\boldsymbol{w}\cdot\boldsymbol{v}$; $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w})=\boldsymbol{u}\cdot\boldsymbol{v}+\boldsymbol{u}\cdot\boldsymbol{w}$; $(k\boldsymbol{v})\cdot\boldsymbol{w}=k(\boldsymbol{v}\cdot\boldsymbol{w})$; $\boldsymbol{v}\cdot\boldsymbol{v}=\lvert\boldsymbol{v}\rvert^2$. Perpendicular vectors; parallel vectors.	Included: for non-zero perpendicular vectors $\boldsymbol{v}\cdot\boldsymbol{w}=0$; for non-zero parallel vectors $\boldsymbol{v}\cdot\boldsymbol{w}=\pm\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert$.	The scalar product is also known as the dot product and the inner product. Link with condition for perpendicularity in §4.3.

4 Core: vector geometry (continued)

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
4.3 The expression $\boldsymbol{v}\cdot\boldsymbol{w}=\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\cos\theta$; the angle between two vectors. The projection of a vector $\boldsymbol{v}$ in the direction of $\boldsymbol{w}$; simple applications, eg finding the distance of a point from a line.	Included: the following formulae $\cos\theta=\dfrac{v_1w_1+v_2w_2}{\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert}$, $\lvert\boldsymbol{v}\rvert\cos\theta=\boldsymbol{v}\cdot\left(\dfrac{\boldsymbol{w}}{\lvert\boldsymbol{w}\rvert}\right)$. Included: an understanding of "$m_1m_2=-1$" $\Rightarrow$ lines are perpendicular.	Link with generalization of perpendicular and parallel cases in §4.2. Application to angle between lines $ax+by=p$ and $cx+dy=q$ as angle between normal vectors. Link with the cosine rule in §3.6.
4.4 The vector product of two vectors $\lvert\boldsymbol{v}\times\boldsymbol{w}\rvert=\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\sin\theta$. The formula for the area of a triangle in the form $\dfrac{1}{2}\lvert\boldsymbol{v}\times\boldsymbol{w}\rvert$.	Included: geometric interpretation of the magnitude of $\boldsymbol{v}\times\boldsymbol{w}$ as the area of a parallelogram. Included: the determinant representation $\boldsymbol{v}\times\boldsymbol{w}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}$.	The vector product is also known as the cross product.
4.5 Vector equation of a line $\boldsymbol{r}=\boldsymbol{a}+\lambda\boldsymbol{b}$. Vector equation of a plane $\boldsymbol{r}=\boldsymbol{a}+\lambda\boldsymbol{b}+\mu\boldsymbol{c}$. Use of normal vector to obtain $\boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}$. Cartesian equations of a line and plane.	Included: cartesian equation of a line in three dimensions $\dfrac{x-x_0}{l}=\dfrac{y-y_0}{m}=\dfrac{z-z_0}{n}$; cartesian equation of a plane $ax+by+cz=d$.	
4.6 Intersections of: two lines; a line with a plane; two planes; three planes. Angle between: two lines; a line and a plane; two planes.	Included: inverse matrix method and Gaussian elimination for finding the intersection of three planes.	Link with solution of linear equations in §5.7.
4.7 Distances in two and three dimensions between points, lines and planes.		

5 Core: matrices and transformations

Teaching time: 20 hours

The aims of this section are to introduce matrices, particularly the algebra of small square matrices, to extend knowledge of transformations, to consider linear transformations of the plane represented by square matrices, to explore composition of transformations and to link matrices to the solution of sets of linear equations.

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
5.1 Definition of a matrix: the terms element, row, column and dimension.		Examples: systems of equations; data storage.
5.2 Algebra of matrices: equality; addition; subtraction; multiplication by a scalar; multiplication of two matrices. The identity matrix.		The matrix facility on a graphic display calculator may be introduced.
5.3 Determinants of matrices; the condition for singularity of a matrix.	Included: matrices of dimension 3×3 at most.	
5.4 The inverse of a square matrix. Inverse of a composite ,$(\boldsymbol{PQ})^{-1} = \boldsymbol{Q}^{-1}\boldsymbol{P}^{-1}$.	Included: matrices of dimension 3×3 at most . Not required: cofactors and minors.	
5.5 Linear transformations of vectors in two dimensions and their matrix representation: rotations; reflections and enlargements. The geometric significance of the determinant.	In examinations: the convention will be that the same symbol will represent both a transformation **and** its matrix, eg $\boldsymbol{R}$ is a rotation of 90° about $(0, 0)$, and $\boldsymbol{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.	Linear transformations are origin invariant. Link with complex numbers in §1.6. Link with transformations of graphs in §2.3.
5.6 Composition of linear transformations $\boldsymbol{P}$, $\boldsymbol{Q}$.	Note that $\boldsymbol{PQ}$ denotes "$\boldsymbol{Q}$ followed by $\boldsymbol{P}$".	
5.7 Solution of linear equations (a maximum of three equations in three unknowns). Conditions for the existence of a unique solution, no solution and an infinity of solutions.		Unique solutions can be found using inverse matrices; other cases using Gaussian elimination. Link with intersections of two lines or three planes in §4.6.

6 Core: statistics

Teaching time: 10 hours

The aims of this section are to explore methods of describing and presenting data, and to introduce methods of measuring central tendency and dispersion of data.

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
6.1 Concept of population and sample. Discrete data and continuous data. Frequency tables.	Included: elementary treatment only.	Data for analysis should be drawn from a wide range of areas.
6.2 Presentation of data. Grouped data; mid-interval values; interval width; upper and lower interval boundaries. Frequency histograms.	Included: treatment of both continuous and discrete data. Note: a frequency histogram uses equal class intervals.	Use of computer spreadsheet software may enhance treatment of this topic.
6.3 Measures of central tendency: sample mean, $\bar{x}$; median.	Included: an awareness that the population mean, μ, is generally unknown, and that the sample mean, $\bar{x}$, serves as an unbiased estimate of this quantity.	
6.4 Cumulative frequency; cumulative frequency graphs; quartiles and percentiles.		Use of box-and-whisker plots on a graphic calculator may enhance understanding.
6.5 Measures of dispersion: range; interquartile range; standard deviation of the sample, s_n. The unbiased estimate, s^2_{n-1}, of the population variance σ^2.	Included: an awareness that the population standard deviation, σ, is generally unknown, and knowledge that $s^2_{n-1} = \frac{n}{n-1} s^2_n$ serves as an unbiased estimate of σ^2. In examinations: candidates are expected to use a statistical function on a calculator to find standard deviations.	Teachers should be aware of calculator, text and regional variations in notation for sample variance.

7 Core: probability

Teaching time: 20 hours

The aims of this section are to extend knowledge of the concepts, notation and laws of probability, and to introduce some important probability distributions and their parameters.

Content	Amplifications/Exclusions	Teaching Notes
7.1 Sample space, U; the event A. The probability of an event A as $\mathrm{P}(A)=\dfrac{n(A)}{n(U)}$. The complementary events A and A' (not A); the relation $\mathrm{P}(A)+\mathrm{P}(A')=1$.	Included: an emphasis on the concept of equally likely outcomes.	Experiments using coins, dice, packs of cards, etc, can enhance understanding of the distinction between (experimental) relative frequency and (theoretical) probability. Simulation using random numbers can also be useful.
7.2 Combined events, $A\cap B$ and $A\cup B$. The relation $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$. Mutually exclusive events; the relation $\mathrm{P}(A\cap B)=0$.	Included: an appreciation of the non-exclusivity of "or".	It should be emphasized that problems might best be solved with the aid of a Venn diagram or tree diagram, without the explicit use of these formulae.
7.3 Conditional probability; the relation $\mathrm{P}(A\mid B)=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$. Independent events; the relations $\mathrm{P}(A\mid B)=\mathrm{P}(A)=\mathrm{P}(A\mid B')$. Use of Bayes' Theorem for two events.	Included: selection without replacement; proof of independence using $\mathrm{P}(A\cap B)=\mathrm{P}(A)\mathrm{P}(B)$.	The term "independent" is equivalent to "statistically independent".
7.4 Use of Venn diagrams, tree diagrams and tables of outcomes to solve problems. Applications.		Examples: cards, dice and other simple cases of random selection.
7.5 Counting principles, including permutations and combinations.	Included: the number of ways of selecting and arranging r objects from n; simple applications.	Link with the binomial theorem in § 1.3.

7 Core: probability (continued)

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
7.6 Discrete probability distributions. Expectation, mode, median, variance and standard deviation.	Included: knowledge and use of the formulae $\mathrm{E}(X)=\sum\left(x\,\mathrm{P}(X=x)\right)$ and $\mathrm{Var}(X)=\mathrm{E}(X-\mu)^2=\mathrm{E}(X^2)-\left[\mathrm{E}(X)\right]^2$.	It is useful to discuss the fact that $\mathrm{E}(X)=0$ indicates a fair game, where X represents the gain of one of the players.
7.7 The binomial distribution, its mean and variance (without proof).	Included: situations and conditions for using a binomial model.	Link with the binomial theorem in §1.3.
7.8 Continuous probability distributions. Expectation, mode, median, variance and standard deviation.	Included: the concept of a continuous random variable; definition and use of probability density functions.	
7.9 The normal distribution. Standardisation of a normal distribution; the use of the standard normal distribution table.	Not included: normal approximation to binomial distribution.	Although candidates will not be expected to use the formula $f(x)=\dfrac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-(x-\mu)^2/2\sigma^2}$, they could be made aware of the fact that it is the probability density function of the normal distribution. Use of calculators as well as tables to find areas and values of z for given probabilities is advised.

8 Core: calculus

Teaching time: 50 hours

The aim of this section is to introduce the basic concepts and techniques of differential and integral calculus, and some of their applications.

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
8.1 Informal ideas of a limit and convergence. The result $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ justified by geometric demonstration.	Included: only a very informal treatment of limit and convergence, eg 0.3, 0.33, 0.333, ... converges to $\frac{1}{3}$.	Link with infinite geometric series in §1.1. Link with the binomial theorem in §1.3. Calculators can be used to investigate limits numerically.
8.2 Differentiation from first principles as the limit of the difference quotient $f'(x) = \lim_{h \to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$ Differentiation of: $x \mapsto x^n, n \in \mathbf{Q}$; $x \mapsto \sin x$; $x \mapsto \cos x$; $x \mapsto \tan x$; $x \mapsto \mathrm{e}^x$; $x \mapsto \ln x$.	Included: a formal treatment for positive integer powers; informal extension to rational powers; a formal treatment for $x \mapsto \sin x$. Included: familiarity with the notation $y = f(x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$.	Other derivatives can be predicted or verified by graphical considerations using graphic display calculator. Investigation of the derivative of x^n from consideration of the function $\frac{(x+h)^n - x^n}{h}$ and its graph, for small h, $n \in \mathbf{Z}^+$, can enhance understanding of limits.
8.3 Differentiation of sums of functions and real multiples of functions. The chain rule for composite functions.	Included: derivatives of reciprocal trigonometric functions $x \mapsto \sec x$, $x \mapsto \csc x$, $x \mapsto \cot x$. Included: applications to rates of change.	Link with composite functions in §2.1. Link with implicit differentiation in §8.7. Link with integration by parts in §8.10.
8.4 Further differentiation: the product and quotient rules; the second derivative; differentiation of a^x and $\log_a x$.	Included: understanding that $2^x = \mathrm{e}^{x\ln 2}$, etc.	

8 Core: calculus (continued)

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
8.5 Graphical behaviour of functions: tangents, normals and singularities, behaviour for large $\lvert x \rvert$; asymptotes. The significance of the second derivative; distinction between maximum and minimum points and points of inflexion.	Included: both "global" and "local" behaviour; choice of appropriate window; (a, b) point of inflexion $\Rightarrow f''(a) = 0$, but the converse is not necessarily true; points of inflexion with zero or non-zero gradient.	Effective use of graphic display calculator envisaged here, combined with sketching by hand. Link with function graphing skills in §2.2. The terms "concave-up" and "concave-down" conveniently distinguish between $f''(x) > 0$ and $f''(x) < 0$ respectively.
8.6 Applications of the first and second derivative to maximum and minimum problems. Kinematic problems involving displacement, s, velocity, $\frac{\mathrm{d}s}{\mathrm{d}t} = v$, and acceleration, $\frac{\mathrm{d}v}{\mathrm{d}t} = a$.	Included: testing for maximum or minimum(eg volume, area and profit) using the sign of the first derivative or using the second derivative.	Link with graphing functions in §2.2.
8.7 Implicit differentiation. Derivatives of the inverse trigonometric functions.	Included: applications to related rates of change. Not required: second derivatives of parametric functions.	Link with chain rule in §8.3.
8.8 Indefinite integration as anti-differentiation. Indefinite integrals of: x^n; $n \in \mathbf{Q}$, $\sin x$; $\cos x$; e^x. Composites of these with $x \mapsto ax + b$. Application to acceleration and velocity.	Included: $\int \frac{1}{x}\,\mathrm{d}x = \ln\lvert x \rvert + C$. Example: $f'(x) = \cos(2x+3)$ $\Rightarrow f(x) = \frac{1}{2}\sin(2x+3) + C$.	Candidates could be made aware of the fundamental theorem of calculus, $F(x) = \int_a^x f(t)\mathrm{d}t \Rightarrow F'(x) = f(x)$, and discuss its graphical interpretation.

8 Core: calculus (continued)

Content	Amplifications/Exclusions	Teaching Notes
8.9 Anti-differentiation with a boundary condition to determine the constant term. Definite integrals. Areas under curves.	Example of a boundary condition: if $\frac{\mathrm{d}s}{\mathrm{d}t} = 3t^2 + t$, and $s = 10$ when $t = 0$, then $s = t^3 + \frac{1}{2}t^2 + 10$.	Area under velocity-time graph representing distance is a useful illustration.
8.10 Further integration: integration by substitution; integration by parts; definite integrals.	Included: limit changes in definite integrals; questions requiring repeated integration by parts; integrals requiring further manipulation, eg, $\int \mathrm{e}^x \sin x \,\mathrm{d}x$; integration using partial fraction decomposition.	Link with transformations of graphs in §2.3. Link with the chain rule in §8.3.
8.11 Solution of first order differential equations by separation of variables.	Included: transformation of a homogeneous equation by the substitution $y = vx$.	Link with exponential and logarithmic functions in §2.10.

9 Option: statistics

Teaching time: 35 hours

The aims of this section are to enable candidates to apply core knowledge of probability distributions and basic statistical calculations, and to make and test hypotheses. A practical approach is envisaged including statistical modelling tasks suitable for inclusion in the portfolio.

Content	Amplifications/Exclusions	Teaching Notes
9.1 Poisson distribution: mean and variance (without proof).	Included: conditions under which a random variable has a Poisson distribution.	Real applications should be introduced, eg the number of telephone calls on a randomly chosen day or the number of cars passing a particular point in an interval.
9.2 Mean and variance of linear combinations of two independent random variables.	Included: $\mathrm{E}(a_1X_1 \pm a_2X_2) = a_1\mathrm{E}(X_1) \pm a_2\mathrm{E}(X_2)$; $\mathrm{Var}(a_1X_1 \pm a_2X_2) = a_1^2\,\mathrm{Var}(X_1) + a_2^2\,\mathrm{Var}(X_2)$.	
9.3 Sampling distribution of the mean. Standard error of the mean. Central limit theorem (without proof). Pooled estimators of population parameters for two samples.	Included: $X \sim \mathrm{N}(\mu, \sigma^2) \Rightarrow \bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$; $\bar{X}$ is approximately $\mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ for large samples whatever the distribution of *X*; for two samples of size *n* and *m*, the pooled unbiased estimators of the population parameters are $\frac{n\bar{X}_n + m\bar{X}_m}{n+m}$, $\frac{nS_n^{\,2} + mS_m^{\,2}}{n+m-2}$.	In some texts and on some calculators the unbiased estimate of the population variance uses alternative notation, eg $\hat{\sigma}^2$, σ_{n-1}^2, σ_x^2, s_x^2.

9 Option: statistics (continued)

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
9.4 Finding confidence intervals for the mean of a normal population from a sample.	Note: if the population variance is known, the normal distribution should be used; if the population variance is unknown, the *t*-distribution should be used (regardless of sample size). On examination papers: the relevant values of the *t*-distribution will be given either in the IBO statistical tables or within the question; alternatively, candidates may use their calculators.	With the advent of statistical software packages and advanced calculator functions, the restriction on the use of *t*-distribution to small samples is no longer necessary. Link with significance testing in §9.5.
9.5 Significance testing: the mean of a sample; the difference between two means. Null and alternative hypotheses H_0 and H_1. Significance levels; critical region and critical values; one-tailed and two-tailed tests. Drawing conclusions.	Use of the normal distribution when σ is known and the *t*-distribution when σ is unknown.	Link with confidence intervals in §9.4. Link with χ^2 distribution in §9.6.
9.6 The χ^2 distribution; degrees of freedom, ν. The χ^2 statistic $\sum \frac{(f_e - f_0)^2}{f_e}$. The χ^2 goodness of fit test.	Included: test for goodness of fit for distributions that could be uniform, binomial, Poisson or normal; the requirement to combine classes with expected frequencies less than five.	Link with significance testing in §9.5.
9.7 Contingency tables. The χ^2 test for the independence of two factors.	Included: Yates' continuity correction for $\nu = 1$.	

10 Option: sets, relations and groups

Teaching time: 35 hours

The aims of this section are to study two important mathematical concepts, sets and groups. The first allows for the extension and development of the notion of a function, while the second provides the framework to discover the common underlying structure unifying many familiar systems.

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
10.1 Finite and infinite sets. Operations on sets: union; intersection; complement. De Morgan's laws; subsets.	Included: illustration of the proof of De Morgan's laws using Venn diagrams.	Examples of set operations on finite and infinite sets will assist understanding.
10.2 Ordered pairs; the cartesian product of two sets. Relations; equivalence relations.	Included: the fact that an equivalence relation on a set induces a partition of the set.	Include examples and visual representations of relations. Link with graphs in §2.2.
10.3 Functions: injections; surjections; bijections. Composition of functions and inverse functions.	Included: knowledge that function composition is not a commutative operation and that if f is a bijection from set A onto set B then f^{-1} exists and is a bijection from set B onto set A.	Link with trigonometric functions in §3.3.
10.4 Binary operations: definition; closure; operation tables.	Note: a binary operation $*$ on a non-empty set S is a rule for combining any two elements $a, b \in S$ to give an element $c \in S$ where $c = a * b$. In examinations: candidates may be required to test whether a given operation satisfies the closure condition.	Examples of binary operations and their closure properties will assist understanding.
10.5 The associative, distributive and commutative properties of binary operations.	Included: the arithmetic operations in $\mathbf{R}$ and $\mathbf{C}$; matrix operations.	Examples of non-commutative operations could be given.

10 Options: sets, relations and groups (continued)

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
10.6 The identity element *e*. The inverse a^{-1} of an element *a*. Proof that the left-cancellation and right-cancellation laws hold, provided the element has an inverse. Proofs of the uniqueness of the identity and inverse elements in particular cases.	Included: knowledge that both the right-identity $a*e=a$ and left-identity $e*a=a$ must hold if *e* is an identity element.	The left-cancellation law is that $a*b=a*c \Rightarrow b=c; \quad a, b, c \in S$. The right-cancellation law is that $b*a=c*a \Rightarrow b=c; \quad a, b, c \in S$.
10.7 The axioms of a group $\{S, *\}$. Abelian groups.	Included: familiarity with a hierarchy of algebraic structures, eg for the set *S* under a given operation the given operation is a binary operation, ie closed, the given operation is associative, an identity element exists under this operation, each element in *S* has an inverse. Note: where the given operation is defined as a a "binary operation", closure may be assumed.	
10.8 Examples of groups: **R**, **Q**, **Z**, and **C** under addition; symmetries of an equilateral triangle and square; matrices of the same order under addition; 2×2 invertible matrices under multiplication; integers under addition modulo *n*; invertible functions under composition of functions; permutations under composition of permutations.	In examinations: for permutations, the form $p=\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ will be used to represent the mapping $1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 2$.	Candidates should be made aware that other forms of notation for permutations will be found in various texts.

10 Option: sets, relations and groups (continued)

Content	Amplifications/Exclusions	Teaching Notes
10.9 Finite and infinite groups. The order of a group element and the order of a group.	Included: an awareness that, in a finite group table, every element appears once only in each row and each column.	
10.10 Cyclic groups and generators of a group. Proof that all cyclic groups are Abelian.	Included: proof that a group of order *n* is cyclic if and only if it contains an element of order *n*.	
10.11 Definition of a subgroup. Lagrange's theorem, without proof, and its corollary.	Included: the test for a subgroup. Note: the corollary of Lagrange's theorem is that the order of the group is divisible by the order of any element.	
10.12 Isomorphism and isomorphic groups: formal definition in terms of a bijection; the property that an isomorphism maps the identity of one group onto the identity of the other group; a similar property for inverses.	Included: isomorphism between two infinite groups. Note: an isomorphism between two groups $(G, *)$ and (H, Δ) is a bijection $\phi: G \to H$ such that $\phi(x * y) = \phi(x) \Delta \phi(y)$; two groups $(G, *)$ and (H, Δ) are isomorphic if there exists an isomorphism for *G* and *H*.	Isomorphism can be demonstrated using the group tables for the following groups: permutations of a set of three elements; symmetries of an equilateral triangle. It may be possible to set up an isomorphism between two groups in more than one way. In any isomorphism between two groups, the corresponding elements must be of the same order.

11 Option: discrete mathematics

Teaching time: 35 hours

The aims of this option are to introduce topics appropriate for the student of mathematics and computer science who will later confront data structures, theory of programming languages and analysis of algorithms, and to explore a variety of applications and techniques of discrete methods and reasoning.

Content	Amplifications/Exclusions	Teaching Notes
11.1 Natural numbers and the well-ordering principle.	Included: knowledge that any non-empty subset of $\mathbf{Z}^+$ contains a smallest element. Included: knowledge that the well-ordering principle implies mathematical induction (without proof).	Recursive definitions and their proofs using mathematical induction could be discussed.
11.2 Division and Euclidean algorithms. The greatest common divisor of integers *a* and *b*, (*a*, *b*). Relatively prime numbers; prime numbers and the fundamental theorem of arithmetic.	Included: the theorem $a \mid b$ and $a \mid c \Rightarrow a \mid (b \pm c)$ and other related theorems; the division algorithm $a = bq + r$ and the Euclidean algorithm for determining the greatest common divisor of two (or more) integers.	Relate to different number systems. If $a, b, c \in \mathbf{Z}^+$, $ax + by = c$ has an integer solution $x = x_0, y = y_0$ if and only if (*a*, *b*) divides *c*. Relate to linear congruence. Proof that the number of primes is infinite is an easy application.
11.3 Congruence modulo *p* as an equivalence class. Residue classes.	Included: relations; equivalence relations; equivalence classes and partitions.	Note that the term residue class is equivalent to congruence class. Link with the division and Euclidean algorithms in §11.2.
11.4 Recurrence relations. Difference equations: basic definitions and solutions of a difference equation.	Included: the equation $Y_{k+1} = AY_k + B$; solutions as sequences; approximating a differential equation by a difference equation; first order difference equations; second order homogeneous difference equations.	Using a difference equation to approximate a differential equation can serve as a good portfolio activity.

11 Option: discrete mathematics (continued)

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
11.5 Simple graphs; connected graphs; complete graphs; multigraphs; directed graphs; bipartite graphs; planar graphs. Subgraphs; complements of graphs. Graph isomorphism.	Included: Euler's relation: $v-e+f=2$; theorems for planar graphs including $e \le 3v-6$, $e \le 2v-4$, κ_5 and $\kappa_{3,3}$ are not planar.	Isomorphism between graphs can be emphasized using a bijection between the vertex sets which preserves adjacency of edges, and using the adjacency matrices of the graphs.
11.6 Walks; Hamiltonian paths and cycles; Eulerian trails and circuits. Graph colouring and chromatic number of a graph.	Included: the following theorems (without proof) a graph is bipartite if and only if $\chi(G)$ is at most 2, if κ_n is a subgraph of *G*, then $\chi(G) \ge n$, if *G* is planar, then $\chi(G) \le 4$ (the 4-colour problem).	$\chi(G)$ is the chromatic number of *G*. Graph colouring is a worthwhile classroom activity
11.7 Networks and trees: definitions and properties. The travelling salesman problem. Rooted trees; binary search trees; weighted trees; sorting; spanning trees; minimal spanning trees. Prim's, Kruskal's and Dijkstra's algorithms.	Included: definitions and examples of depth-first search and breadth-first search algorithms.	Students interested in computing may engage in writing programs for scheduling on a small database. These may include designing transportation networks for a small business, production plans for a product involving several processes. Note that Prim's algorithm is an example of a greedy algorithm since "at each iteration we do the thing that seems best at that step".

12 Option: analysis and approximation

Teaching time: 35 hours

The aims of this section are to use calculus results to solve differential equations (numerically and analytically), to approximate definite integrals, to solve non-linear equations by iteration, and to approximate functions by expansions of power series. The expectation is that candidates will use a graphic display calculator to perform computations and also to develop a sound understanding of the underlying mathematics.

Content	Amplifications/Exclusions	Teaching Notes
12.1 Convergence of infinite series. Tests for convergence: ratio test; limit comparison test; integral test.	Included: conditions for the application of these tests; the divergence theorem, if $\sum u_n$ is a convergent series then $\lim_{n\to\infty} u_n = 0$.	Convergence of an infinite series should be introduced through the convergence of the sequence of partial sums; the limit comparison test and comparison test may then be used.
12.2 Alternating series. Conditional convergence.	Included: knowledge that the absolute value of the truncation error is less than the next term in the series; absolute convergence of an infinite series.	It is useful to explain that $\sum_{1}^{\infty} \frac{1}{n}$ is divergent, but that $\sum_{1}^{\infty} (-1)^n \frac{1}{n}$ is convergent.
12.3 Power series: radius of convergence. Determination of the radius of convergence by the ratio test.	Included: power series in $(x-k)$, $k \neq 0$.	
12.4 Rolle's theorem; the mean value theorem. Applications of these theorems.	Included: graphical representation of these theorems.	Applications of the mean value theorem can include proving inequalities such as $\lvert \sin x - \sin y \rvert < \lvert x - y \rvert$.
12.5 Use of Taylor series expansions, including the error term. Maclaurin series as a special case. Taylor polynomials. Taylor series by multiplication.	Not required: proof of Taylor's theorem. Included: applications to the approximation of functions; bounds on the error term. On examination papers: the form of the error term will be given. Included: finding the Taylor approximations for functions such as $e^{x^2} \arctan x$ by multiplying the Taylor approximations for e^{x^2} and $\arctan x$.	Series expansions for the trigonometric functions and their inverses, and the exponential and logarithmic functions are good examples.

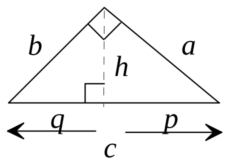
12 Option: analysis and approximation (continued)

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
12.6 Numerical integration. Derivation and application of the trapezium rule and Simpson's rule. The forms of the error terms; their use.	Included: the definition of an integral as the limit of a sum. On examination papers: the forms of the error terms will be given; geometric interpretations will be given.	Comparison of the error estimates for the trapezium rule and Simpson's rule is worthwhile to emphasize the accuracy of the latter.
12.7 The solution of non-linear equations by iterative methods, including the Newton-Raphson method; graphical interpretations. Fixed point iteration; conditions for convergence. The concept of order of convergence (without proof).	Included: choice of initial approximation by the bisection method to solve $f(x)=0$ using the Newton-Raphson method.	The use of a graphic display calculator to help choose a suitable initial approximation to x is valuable, as is the discussion of the calculator algorithms for approximate solution of equations.

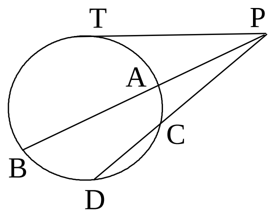
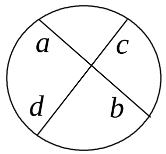
13 Option: Euclidean geometry and conic sections

Teaching time: 35 hours

The aims of this section are to expose candidates to formal proofs in Euclidean geometry thereby providing a broader understanding of the scope of mathematical proof, and to study conic sections using their cartesian equations.

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
13.1 Principles of geometric proof: postulates, theorems and their proof; deductive reasoning; if-then statements and their converses; inductive reasoning; geometric patterns.	Included: use of properties of equivalent (equal area), similar and congruent figures to provide geometric proofs of proportions.	It is helpful to draw comparisons between the reasoning used in proofs in geometry and in other topic areas. Congruence is an equivalence relation.
13.2 Triangles: medians; altitudes; angle bisectors; perpendicular bisectors of sides. Concurrency: orthocentre; incentre; circumcentre; centroid. Principles of construction of triangles from secondary elements using a straight edge and compass. Euler's circle (the nine point circle).	Note: the primary elements of a triangle are the angles and the lengths of the sides; the secondary elements include the altitudes, medians and angle bisectors.	
13.3 Proportional length and proportional division of a line segment (internal and external); the harmonic ratio; proportional segments in right angled triangles. Euclid's theorem for proportional segments in a right angled triangle.	Included: knowledge that the proportional segments p, q satisfy  $h^2 = pq$ $a^2 = pc$ $b^2 = qc$.	This topic can be linked with vector geometry, which provides a useful opportunity to compare different approaches to geometrical proof.

13 Option: Euclidean geometry and conic sections (continued)

CONTENT	AMPLIFICATIONS/EXCLUSIONS	TEACHING NOTES
13.4 Circle geometry: tangents; arcs, chords and secants; the tangent-secant and secant-secant theorems; the intersecting-chords theorem; loci and constructions; inscribed and circumscribed polygons; properties of cyclic quadrilaterals.	Included: the tangent-secant theorem  $\mathrm{PT}^2 = \mathrm{PA} \times \mathrm{PB} = \mathrm{PC} \times \mathrm{PD}$. Included: the intersecting-chords theorem  $ab = cd$. On examination papers: questions will not be set which require constructions with ruler and compasses.	The tangency of two circles and its implications could be discussed.
13.5 Apollonius' theorem (circle of Apollonius); Apollonius' theorem; Menelaus' theorem; Ceva's theorem; Ptolemy's theorem; bisector theorem. Proof of these theorems. The use of the theorems to prove further results.		This sub-topic provides an opportunity to introduce historical connections and the development of the concept of proof. Applications in art and design can be explored.
13.6 Conic sections: focus and directrix; eccentricity. Circle; parabola; hyperbola; ellipse. Parametric equations; the general equation of second degree; rotation of axes.	Included: equations of tangents and normals to these curves; proofs of properties associated with intersections between tangents, normals and curves.	Some exploration of applications in science and industry can enhance understanding. Link with solution of equations of the form $f(x) = g(x)$ in § 2.6.

SYLLABUS GUIDELINES

Presumed knowledge

1 General

Candidates are not required to be familiar with all the topics listed below as presumed knowledge (PK) **before** they start the mathematics HL course of study. However, they should be familiar with these topics before they take the **written papers**, as questions will assume knowledge of them. It is therefore recommended that teachers ensure that any topics from presumed knowledge which are unknown to their candidates at the start of the course are included in the programme of study at an early stage.

Candidates should be familiar with the Système International (SI) units of length, mass and time, and their derived units.

2 Topics

PK1 Number and algebra

1.01 Routine use of addition, subtraction, multiplication and division using integers, decimals and fractions, including order of operations.
Example: $2(3+4\times 7)=62$

1.02 Simple positive exponents.
Examples: $2^3=8;\ (-3)^3=-27;\ (-2)^4=16$

1.03 Simplification of expressions involving roots (surds or radicals).
Examples: $\sqrt{27}+\sqrt{75}=8\sqrt{3};\ \sqrt{3}\times\sqrt{5}=\sqrt{15}$

1.04 Prime numbers and factors, including greatest common factors and least common multiples.

1.05 Simple applications of ratio, percentage and proportion, linked to similarity.

1.06 Definition and elementary treatment of absolute value (modulus), $|a|$.

1.07 Rounding, decimal approximations and significant figures.

1.08 Expression of numbers in standard form (scientific notation), ie, $a \times 10^k$, $1 \le a < 10, k \in \mathbf{Z}$.

1.09 Concept and notation of sets, elements, universal (reference) set, empty (null) set, complement, subset, equality of sets, disjoint sets. Operations on sets: union and intersection. Commutative, associative and distributive properties. Venn diagrams.

1.10 Number systems: natural numbers, $\mathbf{N}$; integers, $\mathbf{Z}$; rationals, $\mathbf{Q}$, and irrationals; real numbers, $\mathbf{R}$.

1.11 Intervals on the real number line using set notation and using inequalities. Expressing the solution set of a linear inequality on the number line and in set notation.

1.12 The concept of a relation between the elements of one set and between the elements of one set and those of another set. Mappings of the elements of one set onto or into another, or the same, set. Illustration by means of tables, diagrams and graphs.

1.13 Basic manipulation of simple algebraic expressions involving factorization and expansion.

Examples: $ab + ac = a(b+c)$; $(a \pm b)^2 = a^2 + b^2 \pm 2ab$;

$a^2 - b^2 = (a-b)(a+b)$; $3x^2 + 5x + 2 = (3x+2)(x+1)$;

$xa - 2a + xb - 2b = (x-2)(a+b)$

1.14 Rearrangement, evaluation and combination of simple formulae. Examples from other subject areas, particularly the sciences, should be included.

1.15 The linear function $x \mapsto ax + b$ and its graph, gradient and y-intercept.

1.16 Addition and subtraction of algebraic fractions with denominators of the form $ax + b$.

Example: $\dfrac{2x}{3x-1} + \dfrac{3x+1}{2x+4}$

1.17 The properties of order relations: $<, \le, >, \ge$.

Examples: $(a > b, c > 0) \Rightarrow ac > bc$; $(a > b, c < 0) \Rightarrow ac < bc$

1.18 Solution of equations and inequalities in one variable including cases with rational coefficients.

Example: $\dfrac{3}{7} - \dfrac{2x}{5} = \dfrac{1}{2}(1-x) \Rightarrow x = \dfrac{5}{7}$

1.19 Solution of $ax^2 + bx + c = 0$, $a \ne 0$. The quadratic formula.

PK2 Geometry

2.01 Elementary geometry of the plane including the concepts of dimension for point, line, plane and space. Parallel and perpendicular lines. Geometry of simple plane figures.

2.02 Angle measurement in degrees. Right-angle trigonometry. Simple applications for solving triangles. Definition of $\tan\theta$ as $\dfrac{\sin\theta}{\cos\theta}$. Graph of $y = x\tan\theta$ with gradient (slope) $\tan\theta$.

2.03 Pythagoras' theorem and its converse.

2.04 The cartesian plane: ordered pairs (x, y), origin, axes. Mid-point of a line segment and distance between two points in the cartesian plane.

2.05 Simple geometric transformations: translation, reflection, rotation, enlargement. Congruence and similarity, including the concept of scale-factor of an enlargement.

2.06 The circle, including arc, chord and tangent properties. Area and circumference.

2.07 Perimeter and area of rectangles, triangles, parallelograms and trapezia (trapezoids), including compound shapes.

PK3 Statistics

3.01 Descriptive statistics: collection of raw data, display of data in pictorial and diagrammatic forms (eg pie charts, pictograms, stem-and-leaf diagrams, bar graphs and line graphs).

3.02 Calculation of simple statistics from discrete data, including mean, median and mode.

Presumed skills

In addition to presumed knowledge, candidates should have the skills to carry out particular mathematical tasks with confidence before starting the course. The course assumes that a candidate will be competent in performing the following basic operations.

- Manipulating indices (exponents) and surds (radicals).
- Solving linear equations and inequalities in one variable, and simultaneous equations in two variables.
- Solving quadratic equations.

- Plotting accurate graphs from a table of values.
- Applying the geometrical properties of the triangle and the circle using the concepts of symmetry, reflection, rotation, similarity and congruence.
- Recognizing, and analysing, the equations of straight lines and circles in the x-y plane. For example, finding points of intersection with axes and determining centres and radii.
- Recognizing quadratic and cubic curves.
- Dealing with errors in numerical calculation due to rounding.
- Applying a sensible degree of accuracy in numerical work.

Internationalism

One of the aims of this course is to enable candidates to appreciate the international dimensions of mathematics and the multiplicity of its cultural and historical perspectives. While this aim is not explicitly written into the syllabus, it is hoped that teachers will take every opportunity to fulfil this aim by discussing relevant issues as they arise and making reference to appropriate background information. For example, it may be appropriate to discuss:

- differences in notation
- the lives of mathematicians set in a historical and/or social context
- the cultural context of mathematical discoveries
- the ways in which certain mathematical discoveries were made in terms of the techniques used
- the attitudinal divergence of different societies towards certain areas of mathematics
- the universality of mathematics as a language.

It should be noted that this aim has not been translated into a corresponding objective. Therefore this aspect of the course will not be tested in examinations.

ASSESSMENT OUTLINE

For first examinations in 2001

External assessment 80%

Written papers 5 hours

Paper 1 2 hours 30%

Twenty compulsory short-response questions based on part I of the syllabus, the compulsory core.

Paper 2 3 hours 50%

Section A: 35%

Five compulsory extended-response questions based on part I of the syllabus, the compulsory core.

Section B: 15%

Five extended-response questions, one on **each** of the optional topics in part II of the syllabus; **one** question to be answered on the chosen topic.

Internal assessment 20%

Portfolio

A collection of **three** pieces of work assigned by the teacher and completed by the candidate during the course. The assignments must be based on different areas of the syllabus and represent all three activities: mathematical investigation; extended closed-problem solving and mathematical modelling.

The portfolio is internally assessed by the teacher and externally moderated by the IBO. Procedures are provided in the *Vade Mecum*.

ASSESSMENT DETAILS

External assessment: written papers

1 General

1 Paper 1 and paper 2

The external assessment consists of two written examination papers, paper 1 and paper 2, which are externally set and externally marked. Together they contribute 80% to the final mark. These papers are designed to allow candidates to demonstrate what they know and can do.

2 Calculators

Candidates are required to have access to a graphic display calculator at all times during the course, both inside and out of the classroom. Regulations concerning the types of calculators allowed are provided in the *Vade Mecum*.

3 Formulae booklet and statistical tables (third edition, February 2001)

As each candidate is required to have access to clean copies of the IBO formulae booklet and statistical tables during the examination, it is recommended that teachers ensure candidates are familiar with the contents of these documents from the beginning of the course. The booklet and tables are provided by IBCA and are published separately.

2 Paper 1: (2 hours) 30%

This paper consists of **twenty** compulsory short-response questions based on part I of the syllabus, the core.

1 Syllabus coverage

- Knowledge of **all** topics from the core is required for this paper.

- The intention of this paper is to test candidates' knowledge across the breadth of the core. However, it should not be assumed that the separate topics from the core will be given equal weight or emphasis.

2 Question type

- A small number of steps will be needed to solve each question.

- Questions may be presented in the form of words, symbols, tables or diagrams, or combinations of these.

3 Mark allocation

- Each question is worth **three** marks. The maximum number of marks available for this paper is **60**, representing 30% of the final assessment.

- Questions of varying levels of difficulty will be set. Each will be worth the same number of marks.

- Full marks are awarded for each **correct** answer irrespective of the presence of working.

  Where a **wrong** answer is given, partial credit may be awarded for a correct method provided this is shown by written working; if no working is present then no partial credit can be given and candidates cannot be awarded any marks. Candidates should therefore be encouraged to show their working at all times.

3 Paper 2: (3 hours) 50%

This paper is divided into two sections: section A, based on part I of the syllabus, and section B, based on part II. It is estimated that, during the total time of three hours, candidates will be able to spend up to 30 minutes in thought and reflection.

1 Question type

- Questions in both sections will require extended responses involving sustained reasoning.

- Individual questions may develop a single theme or be divided into unconnected parts.

- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.

- Normally, each question will reflect an incline of difficulty from relatively easy tasks at the start of a question to relatively difficult tasks at the end of a question. The emphasis will be on problem-solving.

2 Awarding of marks

- Marks will be awarded according to the following categories.

  Method: evidence of knowledge, the ability to apply concepts and skills, and the ability to analyse a problem in a logical manner.

  Accuracy: computational skill and numerical accuracy.

  Reasoning: clear reasoning, explanation and/or logical argument.

  Correct statements: results or conclusions expressed in words.

  Follow through: if an incorrect answer found in an earlier part of a question is used later in the same question then marks may be awarded in the later part even though the original answer used is incorrect. In this way, candidates are not penalized for the same mistake more than once.

- A correct answer with no indication of the method used (for example, in the form of diagrams, graphs, explanations, calculations) will normally be awarded **no** marks. All candidates should therefore be advised to show their working.

4 Paper 2: section A

This section consists of five compulsory extended-response questions based on part I of the syllabus, the core. Candidates will be expected to answer all the questions in this section.

1 Syllabus coverage

- Knowledge of **all** topics from part I of the syllabus is required for this section of paper 2.

- Individual questions may require knowledge of more than one topic from the core.

- The intention of this section is to test candidates' knowledge of the core in depth. A narrower range of topics from the syllabus will be tested in this paper than is tested in paper 1.

2 Mark allocation

- This section is worth **70** marks, representing 35% of the final mark.
- Questions in this section may be unequal in terms of length and level of difficulty. Hence individual questions will not necessarily be worth the same number of marks. The exact number of marks allocated to each question will be indicated at the start of each question.

5 Paper 2: section B

This section consists of five extended-response questions based on part II of the syllabus, the options. **One** question will be set on each option.

1 Syllabus coverage

- Candidates will be expected to answer the question based on the option they have studied.
- Knowledge of the entire contents of the option studied is required for this section of paper 2.
- In order to provide appropriate syllabus coverage of each option, questions in this section are likely to contain two or more unconnected parts.

2 Mark allocation

- This section is worth **30** marks, representing 15% of the final mark.
- Questions in this section will be equal in terms of length and level of difficulty. Each question will be worth 30 marks.

Internal assessment: the portfolio

1 The purpose of the portfolio

The purpose of the portfolio is to provide candidates with opportunities to be rewarded for mathematics carried out under ordinary conditions, that is, without the time limitations and stress associated with written examinations. Consequently the emphasis should be on good mathematical writing and thoughtful reflection.

The portfolio is also intended to provide candidates with opportunities to increase their understanding of mathematical concepts and processes. It is hoped that, in this way, candidates will benefit from these activities and find them both stimulating and rewarding.

The specific purposes of portfolio work are to:

- develop candidates' personal insight into the nature of mathematics and to develop their ability to ask their own questions about mathematics
- provide opportunities for candidates to complete extended pieces of work in mathematics without the time constraints of an examination
- enable candidates to develop individual skills and techniques, and to allow them to experience the satisfaction of applying mathematical processes on their own
- provide candidates with the opportunity to experience for themselves the beauty, power and usefulness of mathematics
- provide candidates with the opportunity to discover, use and appreciate the power of a calculator/computer as a tool for doing mathematics
- enable candidates to develop qualities of patience and persistence, and to reflect on the significance of the results they obtain
- provide opportunities for candidates to show, with confidence, what they know and can do.

2 Requirements

For mathematics HL, the portfolio must consist of a collection of **three pieces of work** assigned by the teacher and completed by the candidate during the course.

Each assignment contained in the portfolio must be based on

- an area of the mathematics HL syllabus
- each of the **three activities**, type I, type II and type III.

The level of sophistication of the mathematics should be about the same as that contained in the syllabus. It is not intended that additional topics be taught to candidates to enable them to complete a particular assignment.

Each portfolio must contain one assignment representing each type of activity. Therefore, the portfolio must contain one type I, one type II and one type III assignment.

- **Type I: mathematical investigation**

  A mathematical investigation is defined as an enquiry into a particular area of mathematics leading to a general result which was previously unknown to the candidate. The use of a calculator and/or computer is encouraged in this type of activity.

  Example: An investigation into the behaviour of the partial sums of a particular sequence.

- **Type II: extended closed-problem solving**

  An extended closed-problem is defined as a multi-part problem where the candidate is guided by developmental questions designed to lead the candidate to a particular result or set of results.

  Example: A group of two or three related questions taken from past examination papers, or a single question with a number of extensions.

- **Type III: mathematical modelling**

  In this context, mathematical modelling is taken to mean the solution of a practical problem set in a real-world context in which the method of solution requires some relatively elementary mathematical modelling skills.

  Example: Analysing the growth of a particular population using a proposed model; reflecting on the nature and usefulness of this model.

3 Integration into the course of study

It is intended that these assignments be completed at intervals throughout the course and not left until towards the end. Indeed, teachers are encouraged to integrate portfolio assignments into their teaching and to allow candidates the opportunity to explore various aspects of as many different topics as possible from both the core and chosen option.

Teachers should not attempt to isolate assignments for the portfolio from what is going on in the classroom, otherwise candidates may regard portfolio assignments as extra work which has to be completed for the sole purpose of the assessment process rather than as a deliberate move to provide them with opportunities for increasing their understanding of mathematical concepts and processes.

Because of the relationship of the portfolio assignments to the syllabus, it is important that each assignment be presented to candidates at the appropriate time. This may be immediately before a topic is introduced, during the study of a topic or immediately after a topic is studied.

Examples:

- A mathematical investigation may be used for the purpose of introducing a topic.
- Mathematical modelling may be used to reinforce mathematical meaning and provide an opportunity for candidates to gain a deeper understanding of the relevant concepts.
- A set of extended closed-problems may be used as a revision exercise.

4 Management of the portfolio

1 Time allocation

The *Vade Mecum* states that a higher level course requires at least **240** teaching hours. In mathematics HL, **10** of these hours should be allocated to work connected to the portfolio. This will allow time for teachers to explain to candidates the requirements of the portfolio and allow class time for candidates to work on individual assignments.

Each assignment should take approximately **three hours** to complete: one hour of class time and two hours of homework time. Consequently, it is expected that during the course candidates will have the time to complete more than three assignments, and will thus be provided with the opportunity to select the best three for inclusion in their portfolios.

For each assignment, class time should be used for candidates to begin (or possibly finish) a piece of work under the guidance of the teacher. It is not intended that this class time be used to introduce material which is not on the syllabus, since each assignment should be based on topics which are within the scope of the syllabus.

2 Setting of assignments

It is the teacher’s responsibility to set suitable assignments which comply with the regulations.

There is no requirement to provide identical assignments for all candidates. Neither is there a requirement to provide each candidate with a different assignment. Teachers may decide which is the best course of action under different circumstances.

Candidates may suggest areas of the syllabus in which they would like to attempt an assignment or may make detailed suggestions as to the form a particular assignment should take. Any such suggestions should be approved by the teacher before work is started.

3 Submission of assignments

The finished piece of work should be submitted to the teacher for assessment soon after it is set, that is, between three and ten days. Candidates should not be given the opportunity to re-submit a piece of work after it has been assessed.

As a guide to length, each piece of work should be approximately equivalent to three or four word-processed pages. However, it should be noted that there is no requirement for work to be word processed.

4 Follow-up and feedback

- Teachers should ensure that candidates are aware of the significance of the results/conclusions which are intended as the outcome of a particular assignment. This is particularly important in the case of investigative work which is used for the purpose of introducing a topic. Some class time devoted to follow-up work should therefore be included when developing a course of study.

- It is also important that candidates receive feedback on their own work so that they are aware of alternative strategies for developing their mathematical thinking and are provided with guidance for improving their skills in writing mathematics.

5 Guidance and authenticity

All candidates should be familiar with the requirements of the portfolio and the means by which it is assessed.

It should be made clear to candidates that writing up a portfolio assignment should be entirely their own work. It is therefore helpful if teachers try to encourage in candidates a sense of responsibility for their own learning so that they accept a degree of ownership and take pride in their own work.

- Time in the classroom can be used for discussion of a particular assignment. This discussion can be between teacher and candidates (or a single candidate) and between two or more candidates. In responding to specific questions from candidates teachers should, where appropriate, guide candidates into more productive routes of enquiry rather than respond with a direct answer.

- When completing a portfolio assignment outside the classroom, candidates should work independently.

Group work, whilst educationally desirable in certain situations, is not appropriate in relation to work being prepared for the portfolio.

Teachers are required to ensure that work submitted for the portfolio is the candidate's own. If in doubt, authenticity may be checked by one or more of the following methods:

- discussion with the candidate
- asking the candidate to explain the methods used and summarise the results
- asking the candidate to repeat the assignment with a different set of data.

It is also appropriate for teachers to request candidates to sign each assignment before submitting it to indicate that it is their own work.

5 Record keeping

Careful records should be kept to ensure that all candidates are able to put together a portfolio which complies with the regulations.

For each assignment, the following should be recorded:

- exact details of the assignment given to the candidate(s)
- areas of the syllabus on which the assignment is based
- the date the assignment was given to the candidate and the date of submission
- the type of activity: type I, type II or type III
- the background to the assignment, in relation to the skills/concepts from the syllabus which had, or had not, been taught to the candidate at the time the assignment was set.

ASSESSMENT CRITERIA

The portfolio

1 Introduction

The portfolio is internally assessed by the teacher and externally moderated by the IBO. Assessment criteria have been developed to address collectively all the group 5 objectives. In developing these criteria, particular attention has been given to the five objectives described below, since these cannot be easily addressed by means of timed written examinations.

Where appropriate in the portfolio, candidates will be expected to:

- organize and present information/data in tabular, graphical and/or diagrammatic forms
- know and use appropriate notation and terminology
- recognize patterns and structures in a variety of situations and draw inductive generalizations
- demonstrate an understanding of, and competence in, the practical applications of mathematics
- use appropriate technological devices as mathematical tools.

2 Form of the assessment criteria

Each piece of work in the portfolio should be assessed against the following four criteria:

A Use of notation and terminology
B Communication
C Mathematical content
D Results and conclusions

In addition, at least one assignment in each portfolio should include work which is appropriate to be assessed against the criterion:

E Making conjectures

And at least one assignment in each portfolio should include work which is appropriate to be assessed against the criterion:

F Use of technology

3 Applying the assessment criteria

The method of assessment used is criterion referenced, not norm referenced. That is, the method of assessing each assignment in a portfolio judges candidates by their performance in relation to identified assessment criteria and not in relation to the work of other candidates.

- Each assignment in the portfolio submitted for mathematics HL is assessed against the four criteria **A** to **D**; at least one is assessed against criterion **E**, and at least one against criterion **F**.

- For each assessment criterion, different levels of achievement are described which concentrate on positive achievement. The description of each achievement level represents the **minimum** requirement for that level to be achieved.

- The aim is to find, for each criterion, the level of achievement gained by the candidate for that piece of work. Consequently, the process involves reading the description of each achievement level, starting with level 0, until one is reached which describes a level of achievement that has **not** been reached. The level of achievement gained by the candidate is therefore the preceding one and it is this which should be recorded.

  For example, if, when considering successive achievement levels for a particular criterion, the description for level 3 does not apply, then level 2 should be recorded.

- If a piece of work appears to fall between two achievement levels then the lower achievement level should be recorded since the minimum requirements for the higher achievement level have not been met.

- For each criterion, only whole numbers may be recorded; fractions and decimals are not acceptable.

- The whole range of achievement levels should be awarded as appropriate. For a particular piece of work, a candidate who attains a high achievement level in relation to one criterion may not necessarily attain high achievement levels in relation to other criteria.

It is recommended that the assessment criteria be made available to candidates at all times.

4 The final mark

Each portfolio should contain three pieces of work. If more than three pieces of work have been completed (this is recommended) then the best three should be included in the portfolio.

To arrive at the final mark for the portfolio:

- **Criteria A–D**: calculate the **average** of all **three** achievement levels for each criterion
- **Criterion E–F**: take the **highest achievement level**

Add these six marks to obtain the final mark, rounding to the nearest integer if necessary. The maximum final mark is 20.

Example The achievement levels for each criterion might be as follows:

Assignment type	Criterion A	Criterion B	Criterion C	Criterion D	Criterion E	Criterion F
I	1	0	2	1	**–**	**–**
II	1	1	3	1	**3**	2
III	2	3	3	1	–	**3**
Final mark	$\mathbf{1\frac{1}{3}}$	$\mathbf{1\frac{1}{3}}$	$\mathbf{2\frac{2}{3}}$	**1**	**3**	**3**

In this case, the final mark would be 12, that is, 12a rounded to the nearest integer since only whole numbers are allowed.

5 Incomplete portfolios

Teachers should ensure that, during the course, all candidates are given the opportunity to complete at least three assignments which comply with the requirements. However, if a candidate’s portfolio contains fewer than three assignments, the final mark should be calculated in exactly the same way as for a complete portfolio, with the missing marks considered to be zeros.

Example In a portfolio which contains only two assignments, the achievement levels for each criterion might be as follows:

Assignment type	Criterion A	Criterion B	Criterion C	Criterion D	Criterion E	Criterion F
I	1	0	2	0	**–**	**–**
II	1	1	3	1	**3**	**–**
III	–	–	–	–	–	–
Final mark	$\mathbf{\frac{2}{3}}$	$\mathbf{\frac{1}{3}}$	$\mathbf{1\frac{2}{3}}$	$\mathbf{\frac{1}{3}}$	**3**	**0**

In this case, the final mark would be 6.

Rounding should only take place at the end of the process in order to obtain the final mark.

6 Achievement Levels

Note that "appropriate" used here means "appropriate to the level of the mathematics HL course".

Criterion A: use of notation and terminology

All three pieces of work in each portfolio should be assessed against this criterion.

Achievement level

0 The candidate **does not use** appropriate notation and terminology.

1 The candidate **uses some** appropriate notation **and/or** terminology.

2 The candidate uses appropriate notation **and** terminology **in a consistent manner and does so throughout the activity**.

Criterion B: communication

All three pieces of work in each portfolio should be assessed against this criterion.

Achievement level

0 The candidate **neither** provides explanations **nor** uses appropriate forms of representation (eg symbols, tables, graphs, diagrams).

1 The candidate **attempts** to provide explanations **and uses some** appropriate forms of representation (eg symbols, tables, graphs, diagrams).

2 The candidate provides **adequate explanations/arguments**, **and communicates** them using appropriate forms of representation (eg symbols, tables, graphs, diagrams).

3 The candidate provides **complete, coherent** explanations/arguments, and communicates them **clearly** using appropriate forms of representation (eg symbols, tables, graphs, diagrams).

Criterion C: mathematical content

All three pieces of work in each portfolio should be assessed against this criterion.

Achievement level

0 The candidate recognizes **no** mathematical concepts which are relevant to the activity.

1 The candidate **recognizes** a mathematical concept **or selects a mathematical strategy which is relevant to the activity**.

2 The candidate recognizes a mathematical concept and **attempts to use** a mathematical strategy which is relevant to the activity **and consistent with the level of the programme**.

3 The candidate recognizes a mathematical concept and **uses** a mathematical strategy which is relevant to the activity and consistent with the level of the programme, and **makes few errors in applying mathematical techniques**.

4 The candidate recognizes a mathematical concept, **successfully** uses a mathematical strategy which is relevant to the activity and consistent with the level of the programme, and applies mathematical techniques **correctly throughout the activity**.

5 The candidate **displays** work distinguished by **precision, insight and a sophisticated level of mathematical understanding**.

Criterion D: results or conclusions

All three pieces of work in each portfolio should be assessed against this criterion. Note that candidates are rewarded for the quality of their conclusions ***or*** *results. This is because most assignments lend themselves to being assessed more appropriately in one or other of these two categories.*

Achievement level

0 The candidate draws **no conclusions** or gives **unreasonable or irrelevant** results.

1 The candidate draws **partial** conclusions or **demonstrates some consideration of the significance or the reasonableness** of results.

2 The candidate draws **adequate** conclusions or demonstrates **some understanding** of the significance **and** reasonableness of results.

3 The candidate draws **full and relevant** conclusions or demonstrates **complete understanding** of the significance, reasonableness or **possible limitations** of results.

Criterion E: making conjectures

A minimum of one piece of work in each portfolio should be assessed against this criterion.

Achievement level	
0	The candidate demonstrates **no awareness** of patterns or structures.
1	The candidate **recognizes** patterns and/or structures.
2	The candidate recognizes patterns and/or structures **and attempts to draw inductive generalizations**.
3	The candidate recognizes patterns and/or structures, **successfully** draws inductive generalizations, **and attempts to provide formal justifications**.
4	The candidate recognizes patterns and/or structures, successfully draws inductive generalizations and **justifies (or disproves) the generalizations by means of formal arguments**.

Criterion F: use of technology

A minimum of one piece of work in each portfolio should be assessed against this criterion.

Achievement level	
0	The candidate **does not use** a calculator or computer beyond routine calculations.
1	The candidate **attempts to use** a calculator or computer **in a manner which could enhance the development of the activity**.
2	The candidate makes **limited** use of a calculator or computer in a manner which **does** enhance the development of the activity.
3	The candidate makes **full and resourceful** use of a calculator or a computer in a manner which **significantly** enhances the development of the activity.

ASSESSMENT GUIDELINES

External assessment: written papers

1 Notation

Of the various notations in use, the IBO has chosen to adopt the notation listed below based on the recommendations of the *International Organization for Standardization*. These will be used on **written examination papers** in mathematics HL without explanation. If forms of notation other than those listed here are used on a particular examination paper then they will be defined within the question in which they appear.

Because candidates are required to recognize, though not necessarily use, the IBO notation in examinations, it is recommended that teachers introduce students to IBO notation at the earliest opportunity. Candidates will **not** be permitted information relating to notation in the examinations.

In a small number of cases, candidates will need to use alternative forms of notation in their written answers as not all forms of IBO notation can be directly transferred into hand-written form. This is true particularly in the case of vectors where the IBO notation uses a bold, italic typeface which cannot be adequately transferred into hand-written form. In this particular case, teachers should advise candidates to use alternative forms of notation in their written work (eg $\vec{x}$, $\bar{x}$ or $\underline{x}$).

$\mathbf{N}$	the set of positive integers and zero, $\{0, 1, 2, 3, ...\}$
$\mathbf{Z}$	the set of integers, $\{0, \pm 1, \pm 2, \pm 3, ...\}$
$\mathbf{Z}^+$	the set of positive integers, $\{1, 2, 3, ...\}$
$\mathbf{Q}$	the set of rational numbers
$\mathbf{Q}^+$	the set of positive rational numbers, $\{x \mid x \in \mathbf{Q}, x > 0\}$
$\mathbf{R}$	the set of real numbers
$\mathbf{R}^+$	the set of positive real numbers, $\{x \mid x \in \mathbf{R}, x > 0\}$
$\mathbf{C}$	the set of complex numbers, $\{a + \mathrm{i}b \mid a, b \in \mathbf{R}\}$
i	$\sqrt{-1}$
z	the complex number $a + \mathrm{i}b = r(\cos\theta + \mathrm{i}\sin\theta)$
z^*	the complex conjugate of z (ie $z^* = a - \mathrm{i}b = r(\cos\theta - \mathrm{i}\sin\theta)$)
$\lvert z \rvert$	the modulus of z
$\arg z$	the argument of z
$\mathrm{Re}\, z$	the real part of z

$\text{Im}\, z$	the imaginary part of z
$\{x_1, x_2, ...\}$	the set with elements $x_1, x_2, ...$
$n(A)$	the number of elements in the finite set A
$\{x \mid \quad\}$	the set of all x such that
$\in$	is an element of
$\notin$	is not an element of
$\varnothing$	the empty (null) set
U	the universal set
$\cup$	union
$\cap$	intersection
$\subset$	is a proper subset of
$\subseteq$	is a subset of
A'	the complement of the set A
$A \times B$	the cartesian product of sets A and B (ie $A \times B = \{(a, b) \mid a \in A, b \in B\}$)
$a^{1/n}$, $\sqrt[n]{a}$	a to the power of $\frac{1}{n}$, nth root of a (if $a \geq 0$ then $\sqrt[n]{a} \geq 0$)
$a^{1/2}$, $\sqrt{a}$	a to the power $\frac{1}{2}$, square root of a (if $a \geq 0$ then $\sqrt{a} \geq 0$)
$\lvert x \rvert$	the modulus or absolute value of x, ie $\begin{cases} x \text{ for } x \geq 0, & x \in \mathbf{R} \\ -x \text{ for } x < 0, & x \in \mathbf{R} \end{cases}$
$\equiv$	identity
$\approx$	is approximately equal to
$>$	is greater than
$\geq$	is greater than or equal to
$<$	is less than
$\leq$	is less than or equal to
$\ngtr$	is not greater than
$\nless$	is not less than
$[a, b]$	the closed interval $a \leq x \leq b$
$]a, b[$	the open interval $a < x < b$
u_n	the nth term of a sequence or series
d	the common difference of an arithmetic sequence
r	the common ratio of a geometric sequence
S_n	the sum of the first n terms of a sequence, $u_1 + u_2 + ... + u_n$
S_∞	the sum to infinity of a sequence, $u_1 + u_2 + ...$
$\sum_{i=1}^{n} u_i$	$u_1 + u_2 + ... + u_n$
$\binom{n}{r}$	$\frac{n!}{r!(n-r)!}$
$f: A \to B$	f is a function under which each element of set A has an image in set B
$f: x \mapsto y$	f is a function under which x is mapped to y
$f(x)$	the image of x under the function f
f^{-1}	the inverse function of the function f
$f \circ g$	the composite function of f and g
$\lim_{x \to a} f(x)$	the limit of $f(x)$ as x tends to a

$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	the derivative of y with respect to x
$f'(x)$	the derivative of $f(x)$ with respect to x
$\int y\,\mathrm{d}x$	the indefinite integral of y with respect to x
$\int_a^b y\,\mathrm{d}x$	the definite integral of y with respect to x between the limits $x = a$ and $x = b$
e^x	exponential function of x
$\log_a x$	logarithm to the base a of x
$\ln x$	the natural logarithm of x, $\log_{\mathrm{e}} x$
$\sin, \cos, \tan$	the circular functions
$\left.\begin{array}{l}\arcsin, \arccos\\ \arctan\end{array}\right\}$	the inverse circular functions
csc, sec, cot	the reciprocal circular functions
$\mathrm{A}(x, y)$	the point A in the plane with cartesian coordinates x and y
[AB]	the line segment with end points A and B
AB	the length of [AB]
(AB)	the line containing points A and B
$\hat{\mathrm{A}}$	the angle at A
$\mathrm{C\hat{A}B}$	the angle between [CA] and [AB]
$\Delta\mathrm{ABC}$	the triangle whose vertices are A, B and C
$\boldsymbol{v}$	the vector $\boldsymbol{v}$
$\overrightarrow{\mathrm{AB}}$	the vector represented in magnitude and direction by the directed line segment from A to B
$\boldsymbol{a}$	the position vector $\overrightarrow{\mathrm{OA}}$
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	unit vectors in the directions of the cartesian coordinate axes
$\lvert\boldsymbol{a}\rvert$	the magnitude of $\boldsymbol{a}$
$\lvert\overrightarrow{\mathrm{AB}}\rvert$	the magnitude of $\overrightarrow{\mathrm{AB}}$
$\boldsymbol{v}\cdot\boldsymbol{w}$	the scalar product of $\boldsymbol{v}$ and $\boldsymbol{w}$
$\boldsymbol{v}\times\boldsymbol{w}$	the vector product of $\boldsymbol{v}$ and $\boldsymbol{w}$
$\boldsymbol{A}^{-1}$	the inverse of the non-singular matrix $\boldsymbol{A}$
$\boldsymbol{A}^{\mathrm{T}}$	the transpose of the matrix $\boldsymbol{A}$
$\det \boldsymbol{A}$	the determinant of the square matrix $\boldsymbol{A}$
$\boldsymbol{I}$	the identity matrix
$\mathrm{P}(A)$	probability of event A
$\mathrm{P}(A')$	probability of the event 'not A'
$\mathrm{P}(A \mid B)$	probability of the event A given B
$x_1, x_2, \ldots$	observations
$f_1, f_2, \ldots$	frequencies with which the observations $x_1, x_2, \ldots$ occur
P_x	probability distribution function $\mathrm{P}(X = x)$ of the discrete random variable X
$f(x)$	probability density function of the continuous random variable X
$\mathrm{E}(X)$	the expected value of the random variable X
$\mathrm{Var}(X)$	the variance of the random variable X
$\mathrm{N}(\mu, \sigma^2)$	normal distribution with mean μ and variance σ^2
$X \sim N(\mu, \sigma^2)$	the random variable X distributed normally with mean μ and variance σ^2

χ^2	the chi-squared distribution
μ	population mean
σ^2	population variance, $\sigma^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n}$, where $n = \sum_{i=1}^{k} f_i$
σ	population standard deviation
$\bar{x}$	sample mean
s_n^2	sample variance, $s_n^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \bar{x})^2}{n}$, where $n = \sum_{i=1}^{k} f_i$
s_n	standard deviation of the sample
s_{n-1}^2	unbiased estimate of the population variance $s_{n-1}^2 = \frac{n}{n-1} s_n^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \bar{x})^2}{n-1}$, where $n = \sum_{i=1}^{k} f_i$
Φ	cumulative distribution function of the standardised normal variable with distribution N(0, 1)
κ_n	a complete graph with n vertices
$\kappa_{n,m}$	a complete bipartite graph with one set of n vertices and another set of m vertices.
$\chi(G)$	the chromatic number of the graph G
$\mathbf{Z}_p$	the set of equivalence classes $\{0, 1, 2, \ldots, p-1\}$ of integers modulo p

2 Terminology (syllabus topic 11, option on discrete mathematics)

Teachers and students should be aware that many different terminologies exist in graph theory and that different textbooks may employ different combinations of these. Examples of these are: vertex/node/junction/point; edge/route/arc; degree of a vertex/order; multiple edges/parallel edges; loop/self-loop.

In IBO examination questions, the terminology used will be as it appears in the syllabus. For clarity these terms are defined below.

- A *graph* consists of a set of *vertices* and a set of *edges*. The endpoints of each edge are connected to either the same vertex or two different vertices.
- An edge whose endpoints are connected to the same vertex is called a *loop*.
- If more than one edge connects the same pair of vertices then these edges are called *multiple edges*.
- A *directed edge* is one in which it is only possible to travel in one direction.
- A *directed graph* is a graph where every edge is directed.
- A *walk* is a sequence of linked edges.
- A *trail* is a walk in which no edge appears more than once.
- A *path* is a walk with no repeated vertices.
- A *circuit* is a walk which begins and ends at the same vertex, and has no repeated edges.
- A *cycle* is a walk which begins and ends at the same vertex, and has no repeated vertices otherwise.
- A *Hamiltonian path* is a path in which all the vertices of a graph appear once.
- A *Hamiltonian cycle* is a path in which all the vertices appear once before it returns to the first vertex.
- A *Eulerian trail* is a trail containing every edge of a graph once.
- A *Eulerian circuit* is a Eulerian trail which begins and ends at the same vertex.
- *Graph colouring* is the assignment of a colour to each vertex in such a way that no two adjacent vertices are assigned the same colour.
- The *chromatic number* of a graph is the minimum number of colours needed to colour the graph.

- The *degree* of a vertex is the number of *edges* connected to that vertex (a loop contributes two, one for each of its endpoints).
- A *simple* graph has no loops or multiple edges.
- A graph is *connected* if there is a path connecting every pair of vertices.
- A graph is *disconnected* if there is at least one pair of vertices which is not connected by a path.
- A *complete* graph is a simple graph, that is, one which has no loops or multiple edges, where every vertex is connected to every other vertex.
- A graph is a *tree* if it is connected and contains no paths which begin and end at the same vertex.
- A *rooted tree* is a directed tree containing a vertex from which there is a path to every other vertex.
- If a rooted tree contains an edge from vertex u to vertex v then u is the *parent* of v and v is the *child* of u.
- A *binary tree* is a rooted tree in which no vertex has more than two children.
- A *binary search tree* is a binary tree in which all children are designated *left* or *right* and no vertex has more than one *left child* or *right child*.
- A *weighted tree* is a tree in which each edge is allocated a number or *weight*.
- *Sorting* is reordering a set into a list of elements in increasing order.
- A *spanning tree* of a graph is a subgraph containing every vertex of the graph, which is also a tree.
- A *minimal spanning tree* is the spanning tree of a weighted graph that has the minimum total weight.
- The *complement* of a graph G is a graph with the same vertices as G but which has an edge between any two vertices if and only if G does not.
- A *graph isomorphism* between two graphs G and H is a one-to-one correspondence between pairs of vertices such that a pair of vertices in G is adjacent if and only if the equivalent pair in H is adjacent.
- A *planar graph* is a graph that can be drawn in the plane without any edge crossing another.
- A *bipartite graph* is a graph whose vertices are divided into two sets and in which edges always connect a vertex from one set to a vertex from the other set.

- A *complete bipartite graph* is a bipartite graph in which there is an edge from every vertex in one set to every vertex in the other set.

- A *subgraph* is a graph within a graph.

- The elements of the *n*th row of an *adjacency matrix* are the number of edges connecting the *n*th vertex with every other vertex, taken in order. Hence, for an undirected graph, the adjacency matrix will be symmetric about the diagonal.

- The elements of the *n*th row of an *incidence matrix* are either 1 or 0 depending on whether each edge, taken in order, is connected to the *n*th vertex or not.

3 Clarification (syllabus topic 13, option on Euclidean geometry and conic sections)

Teachers and students should be aware that some of the theorems mentioned in this section may be known by other names or some names of theorems may be associated with different statements *in some textbooks*. In order to avoid confusion, in IBO examinations, theorems which may be misinterpreted are defined below.

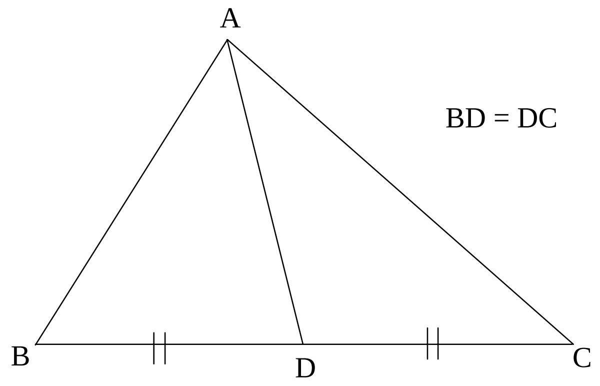
Apollonius' theorem (Circle of Apollonius)

If A, B are two fixed points such that $\frac{\mathrm{PA}}{\mathrm{PB}}$ is a constant not equal to one then the locus of P is a circle. This circle is called the circle of Apollonius.

Remark: the converse of this theorem is included.

Apollonius' theorem

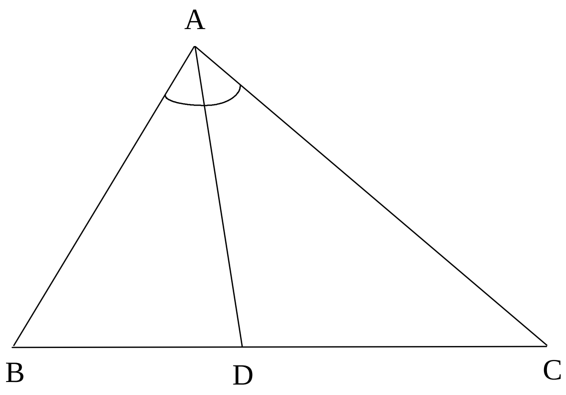
If D is the midpoint of the base [BC] of a triangle ABC, then $\mathrm{AB}^2 + \mathrm{AC}^2 = 2(\mathrm{AD}^2 + \mathrm{BD}^2)$.

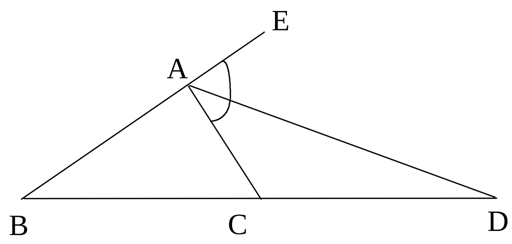


Bisector theorem

The angle bisector of an angle of a triangle divides the side of the triangle opposite the angle into segments proportional to the sides adjacent to the angle.

If ABC is the given triangle with [AD] as the bisector of angle BAC intersecting [BC] at point D, then $\frac{BD}{DC} = \frac{AB}{AC}$.





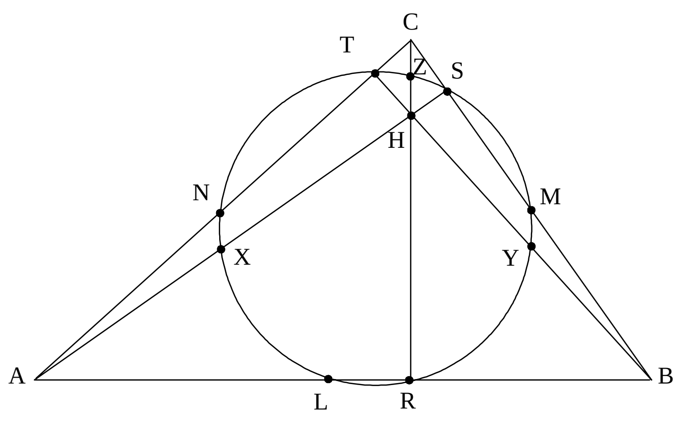
$B\hat{A}D = C\hat{A}D$ $\qquad$ $C\hat{A}D = E\hat{A}D$

Remark: The converse of this result is included.

Nine-point circle theorem

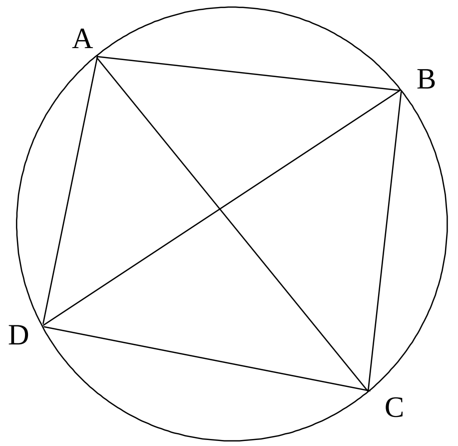
Given any triangle ABC, let H be the intersection of the three altitudes. There is a circle that passes through these nine special points:

the midpoints L, M, N of the three sides
the points R, S, T, where the three altitudes of the triangle meet the sides
the midpoints, X, Y, Z, of [HA], [HB], [HC].



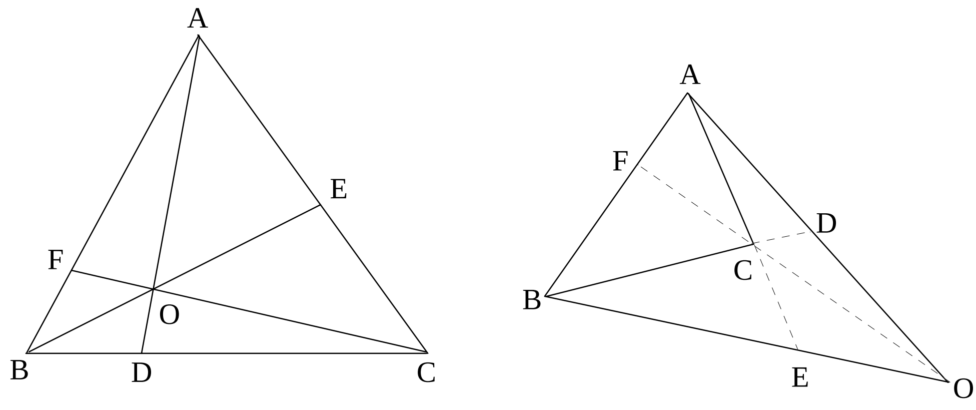
Ptolemy's theorem

If a quadrilateral is cyclic, the sum of the products of the two pairs of opposite sides equals the products of the diagonals, ie for a cyclic quadrilateral ABCD, $\mathrm{AB} \times \mathrm{CD} + \mathrm{BC} \times \mathrm{DA} = \mathrm{AC} \times \mathrm{BD}$.



Ceva's theorem

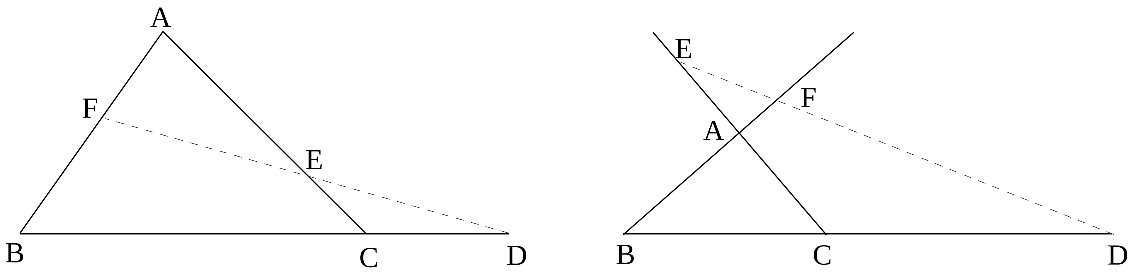
If three concurrent lines are drawn through the vertices A, B, C of a triangle ABC to meet the opposite sides at D, E, F, respectively, then $\frac{\mathrm{BD}}{\mathrm{DC}} \times \frac{\mathrm{CE}}{\mathrm{EA}} \times \frac{\mathrm{AF}}{\mathrm{FB}} = +1$.



Converse: If D, E, F are points on [BC], [CA], [AB], respectively such that $\frac{\mathrm{BD}}{\mathrm{DC}} \times \frac{\mathrm{CE}}{\mathrm{EA}} \times \frac{\mathrm{AF}}{\mathrm{FB}} = +1$, then [AD], [BE] and [CF] are concurrent.

Menelaus' theorem

If a transversal meets the sides [BC], [CA], [AB] of a triangle at D, E, F, respectively, then $\frac{\mathrm{BD}}{\mathrm{DC}} \times \frac{\mathrm{CE}}{\mathrm{EA}} \times \frac{\mathrm{AF}}{\mathrm{FB}} = -1$.



Converse: If D, E, F are points on the sides [BC], [CA], [AB], respectively, of a triangle such that $\frac{\mathrm{BD}}{\mathrm{DC}} \times \frac{\mathrm{CE}}{\mathrm{EA}} \times \frac{\mathrm{AF}}{\mathrm{FB}} = -1$, then D, E, F are collinear.

Note on Ceva's theorem and Menelaus' theorem

The statements and proofs of these theorems presuppose the idea of sensed magnitudes. Two segments [AB], [PQ] of the same or parallel lines are said to have the same sense or opposite senses (or are sometimes called like or unlike) according as the displacements $\mathrm{A} \to \mathrm{B}$ and $\mathrm{P} \to \mathrm{Q}$ are in the same or opposite directions. This may be used to prove the following theorem:

Theorem: If A, B, C are any three collinear points then $\mathrm{AB} + \mathrm{BC} + \mathrm{CA} = 0$ where AB, BC and CA denote sensed magnitudes.

Internal assessment: the portfolio

1 Teaching and learning strategies

- As an integral part of the course, candidates need to be provided with the opportunities to experiment, explore, make conjectures and ask questions. Ideally the atmosphere in the classroom should be one of enquiry.

- It will still be necessary for candidates to learn the skills associated with portfolio activities. One way of approaching this might be for the whole class or smaller groups to work through a small number of relatively simple assignments in order for candidates to be made aware of what might be required, although the work done on these assignments would not be eligible for inclusion in their portfolios.

  For example, candidates may be unaware of certain strategies associated with experimentation, or “playing”, which are an important part of investigative work, particularly if they have only experienced more formal modes of working.

- In reporting their results, candidates should realize that there is an emphasis on thoughtful reflection and good mathematical writing. These are skills that are rarely learned through timed tests/examinations, and therefore candidates may need some guidance and encouragement in these areas.

- It is also important that candidates are given the opportunity to learn mathematical concepts new to them and to gain a deeper understanding of concepts already learned through portfolio activities. It will therefore be necessary to allow time for classroom discussion of the results/conclusions that can be drawn from a particular activity. This time should not be regarded as additional to time allocated to teaching the syllabus since it will, of course, involve discussion of topics which are already part of the syllabus.

2 The nature of portfolio assignments

- Portfolio assignments should provide candidates with opportunities to engage in mathematics in an environment which will capture their interest and provide them with rich opportunities to exercise their mental powers.

- Each assignment should be accessible in terms of the candidates’ background in mathematics and should allow them to achieve results at different levels. This will allow even the weaker candidates to gain a sense of satisfaction in relation to what they have accomplished.

- Assignments should be constructed in a manner which will offer candidates the possibility of gaining the maximum achievement level for each criterion. At the same time, it is accepted that for some activities maximum achievement levels will be more difficult to obtain than for others.

It is therefore important that candidates are presented with a range of activities which will allow them to show what they can do in relation to **each** of the criteria.

- Activities should be presented to candidates in written form. Ideally, all candidates should receive their own copies so that they may make reference to them at any time.

- Within each assignment, the degree to which candidates are guided into choosing particular strategies will depend on the skills the candidate has acquired. Assignments presented to candidates in the earlier stages of the course are therefore likely to be more structured than those presented to candidates towards the end of the course.