International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

Mathematics

Higher level

Specimen papers 1, 2 and 3
(adapted from November 2014)

For first examinations in 2017

CONTENTS

Diploma Programme
Programme du diplôme
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Mathematics
Higher level
Paper 1

SPECIMEN (adapted from November 2014)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[100 marks]**.

12 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

The function f is defined by $f(x)=\dfrac{1}{x}, x \neq 0$.

The graph of the function $y=g(x)$ is obtained by applying the following transformations to the graph of $y=f(x)$:

a translation by the vector $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$;

a translation by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(a) Find an expression for $g(x)$. [2]

(b) State the equations of the asymptotes of the graph of g. [2]

2. [Maximum mark: 6]

The quadratic equation $2x^2 - 8x + 1 = 0$ has roots α and β.

(a) Without solving the equation, find the value of

(i) $\alpha + \beta$;

(ii) $\alpha\beta$. [2]

Another quadratic equation $x^2 + px + q = 0$, $p, q \in \mathbb{Z}$ has roots $\dfrac{2}{\alpha}$ and $\dfrac{2}{\beta}$.

(b) Find the value of p and the value of q. [4]

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3. [Maximum mark: 6]

A point P, relative to an origin O, has position vector $\overrightarrow{\mathrm{OP}} = \begin{pmatrix} 1+s \\ 3+2s \\ 1-s \end{pmatrix}$, $s \in \mathbb{R}$.

(a) Show that $\left|\overrightarrow{\mathrm{OP}}\right|^2 = 6s^2 + 12s + 11$. [1]

(b) Hence find the minimum length of $\overrightarrow{\mathrm{OP}}$. [4]

(c) Explain, geometrically, why your answer gives a minimum value. [1]

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4. [Maximum mark: 7]

Events A and B are such that $\mathrm{P}(A) = 0.2$ and $\mathrm{P}(B) = 0.5$.

(a) Determine the value of $\mathrm{P}(A \cup B)$ when

(i) A and B are mutually exclusive;

(ii) A and B are independent. [4]

(b) Find the smallest and largest possible values of $\mathrm{P}(A \mid B)$. [3]

5. [Maximum mark: 6]

By using the substitution $u = 1 + \sqrt{x}$, find $\int \frac{\sqrt{x}}{1+\sqrt{x}}\,\mathrm{d}x$.

6. [Maximum mark: 7]

Use mathematical induction to prove that $(2n)! \geq 2^n(n!)^2$, $n \in \mathbb{Z}^+$.

7. [Maximum mark: 7]

A continuous random variable T has probability density function f defined by

$$f(t) = \begin{cases} |2-t|, 1 \le t \le 3 \\ 0, \text{ otherwise} . \end{cases}$$

(a) Sketch the graph of $y = f(t)$. [2]

(b) (i) Find the lower quartile of T.

(ii) Hence find the interquartile range of T. [5]

8. [Maximum mark: 7]

A set of positive integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is used to form a pack of nine cards. Each card displays one positive integer without repetition from this set. Grace wishes to select four cards at random from this pack of nine cards.

(a) Find the number of selections Grace could make if the largest integer drawn among the four cards is either a 5, a 6 or a 7. [3]

(b) Find the number of selections Grace could make if at least two of the four integers drawn are even. [4]

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

9. [Maximum mark: 17]

The function f is defined as $f(x) = \mathrm{e}^{3x+1}$, $x \in \mathbb{R}$.

(a) Find $f^{-1}(x)$. [3]

The function g is defined as $g(x) = \ln x$, $x \in \mathbb{R}^+$.
The graph of $y = g(x)$ intersects the x-axis at the point Q.

(b) Show that the equation of the tangent T to the graph of $y = g(x)$ at the point Q is $y = x - 1$. [3]

A region R is bounded by the graphs of $y = g(x)$, the tangent T and the line $x = \mathrm{e}$.

(c) Find the area of the region R. [5]

(d) (i) Show that $g(x) \le x - 1$, $x \in \mathbb{R}^+$.

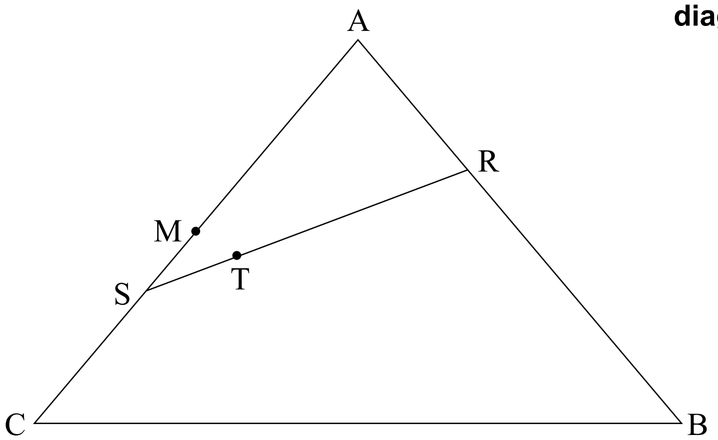
(ii) By replacing x with $\frac{1}{x}$ in part (d)(i), show that $\frac{x-1}{x} \le g(x)$, $x \in \mathbb{R}^+$. [6]

Do **not** write solutions on this page.

10. [Maximum mark: 14]

The position vectors of the points A, B and C are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, relative to an origin O. The following diagram shows the triangle ABC and points M, R, S and T.

diagram not to scale



M is the mid-point of [AC].

R is a point on [AB] such that $\overrightarrow{AR} = \frac{1}{3}\overrightarrow{AB}$.

S is a point on [AC] such that $\overrightarrow{AS} = \frac{2}{3}\overrightarrow{AC}$.

T is a point on [RS] such that $\overrightarrow{RT} = \frac{2}{3}\overrightarrow{RS}$.

(a) (i) Express $\overrightarrow{AM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

(ii) Hence show that $\overrightarrow{BM} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$. [4]

(b) (i) Express $\overrightarrow{RA}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(ii) Show that $\overrightarrow{RT} = -\frac{2}{9}\boldsymbol{a} - \frac{2}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$. [5]

(c) Prove that T lies on [BM]. [5]

Do **not** write solutions on this page.

11. [Maximum mark: 19]

(a) Show that $(1 + \mathrm{i}\tan\theta)^n + (1 - \mathrm{i}\tan\theta)^n = \dfrac{2\cos n\theta}{\cos^n\theta}$, $\cos\theta \neq 0$. [6]

(b) (i) Use the double angle identity $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ to show that $\tan\dfrac{\pi}{8} = \sqrt{2} - 1$.

(ii) Show that $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$.

(iii) Hence find the value of $\displaystyle\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x}\,\mathrm{d}x$. [13]

SPEC/5/MATHL/HP1/ENG/TZ0/XX/M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

Specimen (adapted from November 2014)

Mathematics

Higher level

Paper 1

16 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking May 2016"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685\ldots$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 ***N*** **marks**

Award ***N*** *marks for* ***correct*** *answers where there is* ***no*** *working*.

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 **Implied marks**

Implied marks appear in ***brackets****, for example,* ***(M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 **Follow through marks**

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER… OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14 Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. (a) $g(x) = \frac{1}{x+3} + 1$ ***A1A1***

Note: Award ***A1*** for $x+3$ in the denominator and ***A1*** for the "$+1$".

[2 marks]

(b) $x = -3$ ***A1***
$y = 1$ ***A1***

[2 marks]

Total [4 marks]

2. (a) using the formulae for the sum and product of roots:

(i) $\alpha + \beta = 4$ ***A1***

(ii) $\alpha\beta = \frac{1}{2}$ ***A1***

Note: Award ***A0A0*** if the above results are obtained by solving the original equation (except for the purpose of checking).

[2 marks]

(b) **METHOD 1**

required quadratic is of the form $x^2 - \left(\frac{2}{\alpha} + \frac{2}{\beta}\right)x + \left(\frac{2}{\alpha}\right)\left(\frac{2}{\beta}\right)$ ***(M1)***

$q = \frac{4}{\alpha\beta}$

$q = 8$ ***A1***

$p = -\left(\frac{2}{\alpha} + \frac{2}{\beta}\right)$

$= -\frac{2(\alpha + \beta)}{\alpha\beta}$ ***M1***

$= -\frac{2 \times 4}{\frac{1}{2}}$

$p = -16$ ***A1***

Note: Accept the use of exact roots.

continued…

Question 2 continued

METHOD 2

replacing x with $\frac{2}{x}$ ***M1***

$2\left(\frac{2}{x}\right)^2 - 8\left(\frac{2}{x}\right) + 1 = 0$

$\frac{8}{x^2} - \frac{16}{x} + 1 = 0$ ***(A1)***

$x^2 - 16x + 8 = 0$

$p = -16$ and $q = 8$ ***A1A1***

Note: Award ***A1A0*** for $x^2 - 16x + 8 = 0$ *ie*, if $p = -16$ and $q = 8$ are not explicitly stated.

[4 marks]

Total [6 marks]

3. (a) $\left|\overrightarrow{\mathrm{OP}}\right|^2 = (1+s)^2 + (3+2s)^2 + (1-s)^2$ ***A1***

$= 6s^2 + 12s + 11$ ***AG***

[1 mark]

(b) attempt to differentiate: $\frac{d}{ds}\left|\overrightarrow{\mathrm{OP}}\right|^2$ $(= 12s + 12)$ ***M1***

attempt to solve: $\frac{d}{ds}\left|\overrightarrow{\mathrm{OP}}\right|^2 = 0$ for s ***(M1)***

$s = -1$ ***A1***

the minimum length of $\overrightarrow{\mathrm{OP}}$ is $\sqrt{5}$ ***A1***

[4 marks]

(c) (The point P is restricted to a line)
a line has a unique point of closest approach to the origin, but no maximum ***R1***

[1 mark]

Total [6 marks]

4. (a) (i) use of $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$ ***(M1)***

$\text{P}(A \cup B) = 0.2 + 0.5$

$= 0.7$ ***A1***

(ii) use of $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A)\text{P}(B)$ ***(M1)***

$\text{P}(A \cup B) = 0.2 + 0.5 - 0.1$

$= 0.6$ ***A1***

[4 marks]

(b) $\text{P}(A|B) = \dfrac{\text{P}(A \cap B)}{\text{P}(B)}$

$\text{P}(A|B)$ is a minimum when $A \cap B = \phi$ $\left(\text{P}(A \cap B) = 0\right)$ ***R1***

$\text{P}(A|B)$ is a maximum when $A \subseteq B$ $\left(\text{P}(A \cap B) = \text{P}(A)\right)$ ***R1***

min value $= 0$, max value $= 0.4$ ***A1***

[3 marks]

Total [7 marks]

5. $\dfrac{\text{d}u}{\text{d}x} = \dfrac{1}{2\sqrt{x}}$ ***A1***

$\text{d}x = 2(u-1)\,\text{d}u$

Note: Award the ***A1*** for any correct relationship between $\text{d}x$ and $\text{d}u$.

$\displaystyle\int \frac{\sqrt{x}}{1+\sqrt{x}}\,\text{d}x = 2\int \frac{(u-1)^2}{u}\,\text{d}u$ ***(M1)A1***

Note: Award the ***M1*** for an attempt at substitution resulting in an integral only involving u.

$= 2\displaystyle\int u - 2 + \frac{1}{u}\,\text{d}u$ ***(A1)***

$= u^2 - 4u + 2\ln u (+C)$ ***A1***

$= x - 2\sqrt{x} - 3 + 2\ln\left(1+\sqrt{x}\right)(+C)$ ***A1***

Note: Award the ***A1*** for a correct expression in x, but not necessarily fully expanded/simplified.

[6 marks]

6. let $\mathrm{P}(n)$ be the proposition that $(2n)! \geq 2^n (n!)^2$, $n \in \mathbb{Z}^+$

consider $\mathrm{P}(1)$:

$2! = 2$ and $2^1(1!)^2 = 2$ so $\mathrm{P}(1)$ is true **R1**

assume $\mathrm{P}(k)$ is true *ie* $(2k)! \geq 2^k (k!)^2$, $k \in \mathbb{Z}^+$ **M1**

Note: Do not award ***M1*** for statements such as "let $n = k$ ".

consider $\mathrm{P}(k+1)$:

$\left(2(k+1)\right)! = (2k+2)(2k+1)(2k)!$ **M1**

$\left(2(k+1)\right)! \geq (2k+2)(2k+1)(k!)^2 2^k$ **A1**

Note: Condone "working backwards" up to this point, but no further unless it is fully justified.

$= 2(k+1)(2k+1)(k!)^2 2^k$

$> 2^{k+1}(k+1)(k+1)(k!)^2$ since $2k+1 > k+1$ **R1**

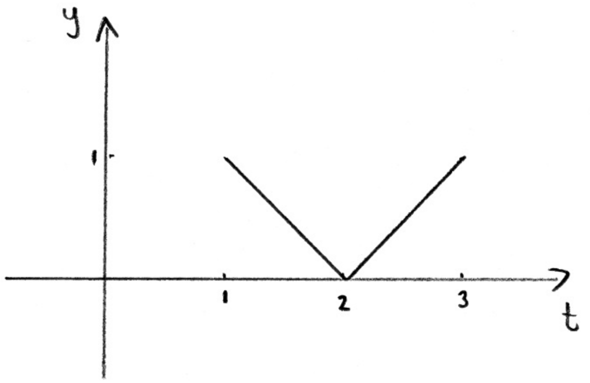
$= 2^{k+1}\left((k+1)!\right)^2$ **A1**

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ **R1**

Note: To obtain the final ***R1***, four of the previous marks must have been awarded.

[7 marks]

7. (a)



$|2-t|$ correct for $[1, 2]$ **A1**

$|2-t|$ correct for $[2, 3]$ **A1**

[2 marks]

(b) (i) let q_1 be the lower quartile

consider $\int_1^{q_1} (2-t)\,dt = \frac{1}{4}$ **M1A1**

obtain $q_1 = 2 - \frac{1}{\sqrt{2}}$ **A1**

continued…

Question 7 continued

(ii) by symmetry, for example, $q_3 = 2 + \frac{1}{\sqrt{2}}$ ***A1***

hence $\text{IQR} = \sqrt{2}$ ***A1***

Note: Only accept this final answer for the ***A1***.

[5 marks]

Total [7 marks]

8. (a) use of the addition principle with 3 terms ***(M1)***

to obtain ${}^4C_3 + {}^5C_3 + {}^6C_3$ $(= 4 + 10 + 20)$ ***A1***

number of possible selections is 34 ***A1***

[3 marks]

(b) **EITHER**

recognition of three cases: (2 odd and 2 even or 1 odd and 3 even or 0 odd and 4 even) ***(M1)***

$\left({}^5C_2 \times {}^4C_2\right) + \left({}^5C_1 \times {}^4C_3\right) + \left({}^5C_0 \times {}^4C_4\right)$ $(= 60 + 20 + 1)$ ***(M1)A1***

OR

recognition to subtract the sum of 4 odd and 3 odd and 1 even from the total ***(M1)***

${}^9C_4 - {}^5C_4 - \left({}^5C_3 \times {}^4C_1\right)$ $(= 126 - 5 - 40)$ ***(M1)A1***

THEN

number of possible selections is 81 ***A1***

[4 marks]

Total [7 marks]

Section B

9. (a) $x = \mathrm{e}^{3y+1}$ ***M1***

Notes: The ***M1*** is for switching variables and can be awarded at any stage. Further marks do not rely on this mark being awarded.

taking the natural logarithm of both sides and attempting to transpose ***M1***

$\left(f^{-1}(x)\right) = \frac{1}{3}(\ln x - 1)$ ***A1***

[3 marks]

(b) coordinates of Q are $(1, 0)$ seen anywhere ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$ ***M1***

at Q, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ ***A1***

$y = x - 1$ ***AG***

[3 marks]

(c) let the required area be A

$A = \int_1^e x - 1\mathrm{d}x - \int_1^e \ln x\mathrm{d}x$ ***M1***

Notes: The ***M1*** is for a difference of integrals. Condone absence of limits here.

attempting to use integration by parts to find $\int \ln x\mathrm{d}x$ ***(M1)***

$= \left[\frac{x^2}{2} - x\right]_1^{\mathrm{e}} - [x\ln x - x]_1^{\mathrm{e}}$ ***A1A1***

Note: Award ***A1*** for $\frac{x^2}{2} - x$ and ***A1*** for $x\ln x - x$.

Note: The second ***M1*** and second ***A1*** are independent of the first ***M1*** and the first ***A1***.

$= \frac{\mathrm{e}^2}{2} - \mathrm{e} - \frac{1}{2}\left(= \frac{\mathrm{e}^2 - 2\mathrm{e} - 1}{2}\right)$ ***A1***

[5 marks]

continued…

Question 9 continued

(d) (i) **METHOD 1**

consider for example $h(x) = x - 1 - \ln x$

$h(1) = 0$ and $h'(x) = 1 - \dfrac{1}{x}$ ***(A1)***

as $h'(x) \geq 0$ for $x \geq 1$, then $h(x) \geq 0$ for $x \geq 1$ ***R1***

as $h'(x) \leq 0$ for $0 < x \leq 1$, then $h(x) \geq 0$ for $0 < x \leq 1$ ***R1***

so $g(x) \leq x - 1$, $x \in \mathbb{R}^+$ ***AG***

METHOD 2

$g''(x) = -\dfrac{1}{x^2}$ ***A1***

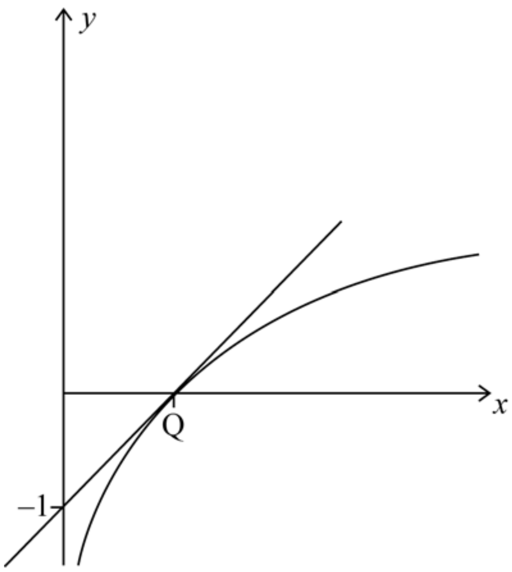
$g''(x) < 0$ (concave down) for $x \in \mathbb{R}^+$ ***R1***

the graph of $y = g(x)$ is below its tangent ($y = x - 1$ at $x = 1$) ***R1***

so $g(x) \leq x - 1$, $x \in \mathbb{R}^+$ ***AG***

Note: The reasoning may be supported by drawn graphical arguments.

METHOD 3



clear correct graphs of $y = x - 1$ and $\ln x$ for $x > 0$ ***A1A1***

statement to the effect that the graph of $\ln x$ is below the graph of its tangent at $x = 1$ ***R1AG***

continued…

Question 9 continued

(ii) replacing x by $\frac{1}{x}$ to obtain $\ln\left(\frac{1}{x}\right) \le \frac{1}{x} - 1 \left(= \frac{1-x}{x}\right)$ ***M1***

$-\ln x \le \frac{1}{x} - 1 \left(= \frac{1-x}{x}\right)$ ***(A1)***

$\ln x \ge 1 - \frac{1}{x} \left(= \frac{x-1}{x}\right)$ ***A1***

so $\frac{x-1}{x} \le g(x)$, $x \in \mathbb{R}^+$ ***AG***

[6 marks]

Total [17 marks]

10. (a) (i) $\overrightarrow{\mathrm{AM}} = \frac{1}{2}\overrightarrow{\mathrm{AC}}$ ***(M1)***

$= \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$ ***A1***

(ii) $\overrightarrow{\mathrm{BM}} = \overrightarrow{\mathrm{BA}} + \overrightarrow{\mathrm{AM}}$ ***M1***

$= \boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$ ***A1***

$\overrightarrow{\mathrm{BM}} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$ ***AG***

[4 marks]

(b) (i) $\overrightarrow{\mathrm{RA}} = \frac{1}{3}\overrightarrow{\mathrm{BA}}$

$= \frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$ ***A1***

(ii) $\overrightarrow{\mathrm{RT}} = \frac{2}{3}\overrightarrow{\mathrm{RS}}$

$= \frac{2}{3}\left(\overrightarrow{\mathrm{RA}} + \overrightarrow{\mathrm{AS}}\right)$ ***(M1)***

$= \frac{2}{3}\left(\frac{1}{3}(\boldsymbol{a} - \boldsymbol{b}) + \frac{2}{3}(\boldsymbol{c} - \boldsymbol{a})\right)$ or equivalent ***A1A1***

$= \frac{2}{9}(\boldsymbol{a} - \boldsymbol{b}) + \frac{4}{9}(\boldsymbol{c} - \boldsymbol{a})$ ***A1***

$\overrightarrow{\mathrm{RT}} = -\frac{2}{9}\boldsymbol{a} - \frac{2}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$ ***AG***

[5 marks]

continued…

Question 10 continued

(c) $\overrightarrow{\mathrm{BT}} = \overrightarrow{\mathrm{BR}} + \overrightarrow{\mathrm{RT}}$

$= \frac{2}{3}\overrightarrow{\mathrm{BA}} + \overrightarrow{\mathrm{RT}}$ ***(M1)***

$= \frac{2}{3}\boldsymbol{a} - \frac{2}{3}\boldsymbol{b} - \frac{2}{9}\boldsymbol{a} - \frac{2}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$ ***A1***

$\overrightarrow{\mathrm{BT}} = \frac{8}{9}\left(\frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)$ ***A1***

point B is common to $\overrightarrow{\mathrm{BT}}$ and $\overrightarrow{\mathrm{BM}}$ and $\overrightarrow{\mathrm{BT}} = \frac{8}{9}\overrightarrow{\mathrm{BM}}$ ***R1R1***

so T lies on [BM] ***AG***

[5 marks]

Total [14 marks]

11. (a) **METHOD 1**

$(1 + \mathrm{i}\tan\theta)^n + (1 - \mathrm{i}\tan\theta)^n = \left(1 + \mathrm{i}\frac{\sin\theta}{\cos\theta}\right)^n + \left(1 - \mathrm{i}\frac{\sin\theta}{\cos\theta}\right)^n$ ***M1***

$= \left(\frac{\cos\theta + \mathrm{i}\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos\theta - \mathrm{i}\sin\theta}{\cos\theta}\right)^n$ ***A1***

by de Moivre's theorem ***(M1)***

$\left(\frac{\cos\theta + \mathrm{i}\sin\theta}{\cos\theta}\right)^n = \frac{\cos n\theta + \mathrm{i}\sin n\theta}{\cos^n\theta}$ ***A1***

recognition that $\cos\theta - \mathrm{i}\sin\theta$ is the complex conjugate of $\cos\theta + \mathrm{i}\sin\theta$ ***(R1)***

use of the fact that the operation of complex conjugation commutes with the operation of raising to an integer power

$\left(\frac{\cos\theta - \mathrm{i}\sin\theta}{\cos\theta}\right)^n = \frac{\cos n\theta - \mathrm{i}\sin n\theta}{\cos^n\theta}$ ***A1***

$(1 + \mathrm{i}\tan\theta)^n + (1 - \mathrm{i}\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}$ ***AG***

continued…

Question 11 continued

METHOD 2

$(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=(1+\mathrm{i}\tan\theta)^n+(1+\mathrm{i}\tan(-\theta))^n$ ***(M1)***

$=\frac{(\cos\theta+\mathrm{i}\sin\theta)^n}{\cos^n\theta}+\frac{(\cos(-\theta)+\mathrm{i}\sin(-\theta))^n}{\cos^n\theta}$ ***M1A1***

Note: Award ***M1*** for converting to cosine and sine terms.

use of de Moivre's theorem ***(M1)***

$=\frac{1}{\cos^n\theta}(\cos n\theta+\mathrm{i}\sin n\theta+\cos(-n\theta)+\mathrm{i}\sin(-n\theta))$ ***A1***

$=\frac{2\cos n\theta}{\cos^n\theta}$ as $\cos(-n\theta)=\cos n\theta$ and $\sin(-n\theta)=-\sin n\theta$ ***R1AG***

[6 marks]

(b) (i) $\tan\frac{\pi}{4}=\frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$ ***(M1)***

$\tan^2\frac{\pi}{8}+2\tan\frac{\pi}{8}-1=0$ ***A1***

let $t=\tan\frac{\pi}{8}$

attempting to solve $t^2+2t-1=0$ for t ***M1***

$t=-1\pm\sqrt{2}$ ***A1***

$\frac{\pi}{8}$ is a first quadrant angle and tan is positive in this quadrant,

so $\tan\frac{\pi}{8}>0$ ***R1***

$\tan\frac{\pi}{8}=\sqrt{2}-1$ ***AG***

(ii) $\cos 4x=2\cos^2 2x-1$ ***A1***

$=2(2\cos^2 x-1)^2-1$ ***M1***

$=2(4\cos^4 x-4\cos^2 x+1)-1$ ***A1***

$=8\cos^4 x-8\cos^2 x+1$ ***AG***

Note: Accept equivalent complex number derivation.

continued…

Question 11 continued

(iii) $\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x}\,dx = 2\int_0^{\frac{\pi}{8}} \frac{8\cos^4 x - 8\cos^2 x + 1}{\cos^2 x}\,dx$

$= 2\int_0^{\frac{\pi}{8}} 8\cos^2 x - 8 + \sec^2 x\,dx$ ***M1***

Note: The ***M1*** is for an integrand involving no fractions.

use of $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$ ***M1***

$= 2\int_0^{\frac{\pi}{8}} 4\cos 2x - 4 + \sec^2 x\,dx$ ***A1***

$= \left[4\sin 2x - 8x + 2\tan x\right]_0^{\frac{\pi}{8}}$ ***A1***

$= 4\sqrt{2} - \pi - 2$ (or equivalent) ***A1***

[13 marks]

Total [19 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 2

SPECIMEN (adapted from November 2014)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[100 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

Consider the two planes

$$\pi_1 : 4x + 2y - z = 8$$

$$\pi_2 : x + 3y + 3z = 3.$$

Find the angle between π_1 and π_2, giving your answer correct to the nearest degree.

2. [Maximum mark: 6]

The wingspans of a certain species of bird can be modelled by a normal distribution with mean 60.2 cm and standard deviation 2.4 cm.

According to this model, 99% of wingspans are greater than x cm.

(a) Find the value of x. [2]

In a field experiment, a research team studies a large sample of these birds. The wingspans of each bird are measured correct to the nearest 0.1 cm.

(b) Find the probability that a randomly selected bird has a wingspan measured as 60.2 cm. [4]

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3. [Maximum mark: 8]

The lines l_1 and l_2 are defined as

$$l_1 : \frac{x-1}{3} = \frac{y-5}{2} = \frac{z-12}{-2}$$

$$l_2 : \frac{x-1}{8} = \frac{y-5}{11} = \frac{z-12}{6}.$$

The plane π contains both l_1 and l_2.

(a) Find the Cartesian equation of π. [4]

The line l_3 passing through the point $(4, 0, 8)$ is perpendicular to π.

(b) Find the coordinates of the point where l_3 meets π. [4]

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4. [Maximum mark: 6]

Consider $p(x) = 3x^3 + ax + 5a$, $a \in \mathbb{R}$.

The polynomial $p(x)$ leaves a remainder of -7 when divided by $(x - a)$.

Show that only one value of a satisfies the above condition and state its value.

5. [Maximum mark: 9]

The seventh, third and first terms of an arithmetic sequence form the first three terms of a geometric sequence.

The arithmetic sequence has first term a and non-zero common difference d.

(a) Show that $d = \frac{a}{2}$. [3]

The seventh term of the arithmetic sequence is 3. The sum of the first n terms in the arithmetic sequence exceeds the sum of the first n terms in the geometric sequence by at least 200.

(b) Find the least value of n for which this occurs. [6]

6. [Maximum mark: 7]

A particle moves in a straight line such that its velocity, $v\,\mathrm{m\,s^{-1}}$, at time t seconds, is given by

$$v(t)=\begin{cases} 5-(t-2)^2, & 0\le t\le 4 \\ 3-\dfrac{t}{2}, & t>4 \end{cases}.$$

(a) Find the value of t when the particle is instantaneously at rest. [2]

The particle returns to its initial position at $t=T$.

(b) Find the value of T. [5]

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7. [Maximum mark: 8]

Compactness is a measure of how compact an enclosed region is.

The compactness, C, of an enclosed region can be defined by $C = \frac{4A}{\pi d^2}$, where A is the area of the region and d is the maximum distance between any two points in the region.

For a circular region, $C = 1$.

Consider a regular polygon of n sides constructed such that its vertices lie on the circumference of a circle of diameter x units.

(a) If $n > 2$ and even, show that $C = \frac{n}{2\pi}\sin\frac{2\pi}{n}$. [3]

If $n > 1$ and odd, it can be shown that $C = \frac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)}$.

(b) Find the regular polygon with the least number of sides for which the compactness is more than 0.99. [4]

(c) Comment briefly on whether C is a good measure of compactness. [1]

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

8. [Maximum mark: 12]

Consider the triangle PQR where $\mathrm{Q\hat{P}R} = 30^\circ$, $\mathrm{PQ} = (x+2)\,\mathrm{cm}$ and $\mathrm{PR} = (5-x)^2\,\mathrm{cm}$, where $-2 < x < 5$.

(a) Show that the area, $A\ \mathrm{cm}^2$, of the triangle is given by $A = \frac{1}{4}\left(x^3 - 8x^2 + 5x + 50\right)$. [2]

(b) (i) State $\frac{\mathrm{d}A}{\mathrm{d}x}$.

(ii) Verify that $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ when $x = \frac{1}{3}$. [3]

(c) (i) Find $\frac{\mathrm{d}^2 A}{\mathrm{d}x^2}$ and hence justify that $x = \frac{1}{3}$ gives the maximum area of triangle PQR.

(ii) State the maximum area of triangle PQR.

(iii) Find QR when the area of triangle PQR is a maximum. [7]

Do **not** write solutions on this page.

9. [Maximum mark: 10]

The number of complaints per day received by customer service at a department store follows a Poisson distribution with a mean of 0.6.

(a) On a randomly chosen day, find the probability that

(i) there are no complaints;

(ii) there are at least three complaints. [3]

(b) In a randomly chosen five-day week, find the probability that there are no complaints. [2]

(c) On a randomly chosen day, find the most likely number of complaints received. Justify your answer. [3]

The department store introduces a new policy to improve customer service. The number of complaints received per day now follows a Poisson distribution with mean λ.

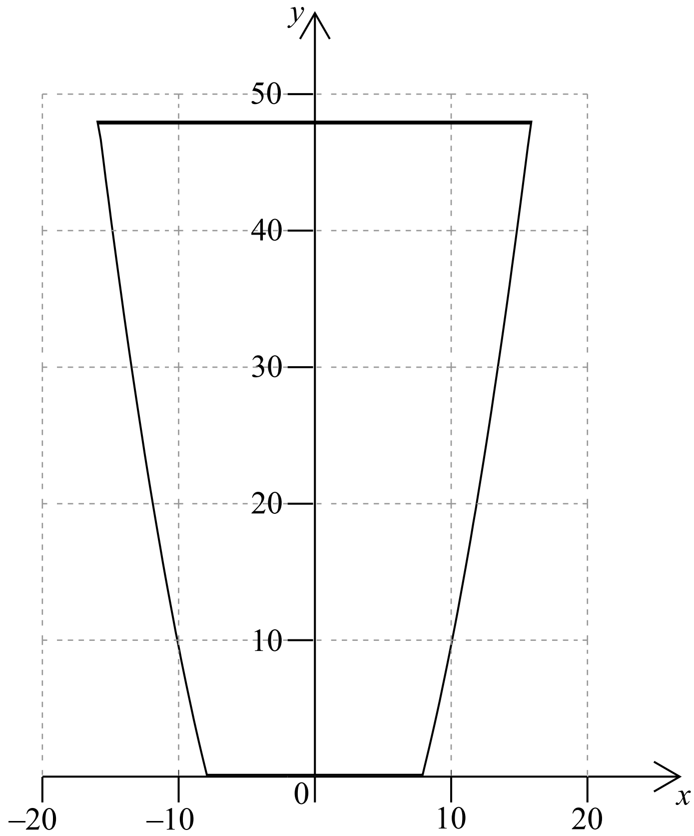
On a randomly chosen day, the probability that there are no complaints is now 0.8.

(d) Find the value of λ. [2]

Do **not** write solutions on this page.

10. [Maximum mark: 17]

The vertical cross-section of a container is shown in the following diagram.



The curved sides of the cross-section are given by the equation $y = 0.25x^2 - 16$.
The horizontal cross-sections are circular. The depth of the container is $48\,\mathrm{cm}$.

(a) If the container is filled with water to a depth of $h\ \mathrm{cm}$, show that the volume, $V\ \mathrm{cm}^3$, of the water is given by $V = 4\pi\left(\dfrac{h^2}{2} + 16h\right)$. [3]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

The container, initially full of water, begins leaking from a small hole at a rate given by $\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{250\sqrt{h}}{\pi(h+16)}$ where t is measured in seconds.

(b) (i) Show that $\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$.

(ii) State $\frac{\mathrm{d}t}{\mathrm{d}h}$ and hence show that $t = \frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)\mathrm{d}h$.

(iii) Find, correct to the nearest minute, the time taken for the container to become empty. (60 seconds = 1 minute) [11]

Once empty, water is pumped back into the container at a rate of $8.5\,\mathrm{cm^3\,s^{-1}}$. At the same time, water continues leaking from the container at a rate of $\frac{250\sqrt{h}}{\pi(h+16)}\mathrm{cm^3\,s^{-1}}$.

(c) Using an appropriate sketch graph, determine the depth at which the water ultimately stabilizes in the container. [3]

Do **not** write solutions on this page.

11. [Maximum mark: 11]

In triangle ABC,

$$3\sin B + 4\cos C = 6 \text{ and}$$
$$4\sin C + 3\cos B = 1\,.$$

(a) Show that $\sin(B+C) = \dfrac{1}{2}$. [6]

Robert conjectures that $\mathrm{C\hat{A}B}$ can have two possible values.

(b) Show that Robert's conjecture is incorrect by proving that $\mathrm{C\hat{A}B}$ has only one possible value. [5]

Please **do not** write on this page.

Answers written on this page will not be marked.

Please **do not** write on this page.

Answers written on this page will not be marked.

Markscheme

Specimen (adapted from November 2014)

Mathematics

Higher level

Paper 2

16 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking May 2016"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685\ldots$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER… OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14 Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. $\boldsymbol{n}_1 = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$ ***(A1)(A1)***

use of $\cos\theta = \dfrac{\boldsymbol{n}_1 \bullet \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$ ***(M1)***

$\cos\theta = \dfrac{7}{\sqrt{21}\sqrt{19}} \left(= \dfrac{7}{\sqrt{399}} \right)$ ***(A1)(A1)***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$\theta = 69°$ ***A1***

Note: Award ***A1*** for $111°$.

[6 marks]

2. (a) $\mathrm{P}(X > x) = 0.99 \quad (= \mathrm{P}(X < x) = 0.01)$ ***(M1)***

$\Rightarrow x = 54.6\,(\mathrm{cm})$ ***A1***

[2 marks]

(b) $\mathrm{P}(60.15 \le X \le 60.25)$ ***(M1)(A1)(A1)***

$= 0.0166$ ***A1***

[4 marks]

Total [6 marks]

3. (a) attempting to find a normal to π *eg* $\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 8 \\ 11 \\ 6 \end{pmatrix}$ ***(M1)***

$\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 8 \\ 11 \\ 6 \end{pmatrix} = 17 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ ***(A1)***

$\boldsymbol{r} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 12 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ ***M1***

$2x - 2y + z = 4$ (or equivalent) ***A1***

[4 marks]

continued…

Question 3 continued

(b) $l_3 : \boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} + t\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, t \in \mathbb{R}$ ***(A1)***

attempting to solve $\begin{pmatrix} 4+2t \\ -2t \\ 8+t \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 4$ for t *ie* $9t + 16 = 4$ for t ***M1***

$t = -\frac{4}{3}$ ***A1***

$\left(\frac{4}{3}, \frac{8}{3}, \frac{20}{3}\right)$ ***A1***

[4 marks]

Total [8 marks]

4. using $p(a) = -7$ to obtain $3a^3 + a^2 + 5a + 7 = 0$ ***M1A1***

$(a+1)(3a^2 - 2a + 7) = 0$ ***(M1)(A1)***

Note:	Award ***M1*** for a cubic graph with correct shape and ***A1*** for clearly showing that the above cubic crosses the horizontal axis at $(-1, 0)$ only.

$a = -1$ ***A1***

EITHER

showing that $3a^2 - 2a + 7 = 0$ has no real (two complex) solutions for a ***R1***

OR

showing that $3a^3 + a^2 + 5a + 7 = 0$ has one real (and two complex) solutions for a ***R1***

Note:	Award ***R1*** for solutions that make specific reference to an appropriate graph.

Total [6 marks]

5. (a) using $r = \frac{u_2}{u_1} = \frac{u_3}{u_2}$ to form $\frac{a + 2d}{a + 6d} = \frac{a}{a + 2d}$ ***(M1)***

$a(a + 6d) = (a + 2d)^2$ ***A1***

$2d(2d - a) = 0$ (or equivalent) ***A1***

since $d \neq 0 \Rightarrow d = \frac{a}{2}$ ***AG***

[3 marks]

(b) substituting $d = \frac{a}{2}$ into $a + 6d = 3$ and solving for a and d ***(M1)***

$a = \frac{3}{4}$ and $d = \frac{3}{8}$ ***(A1)***

$r = \frac{1}{2}$ ***A1***

$$\frac{n}{2}\left(2 \times \frac{3}{4} + (n - 1)\frac{3}{8}\right) - \frac{3\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} \geq 200$$ ***(A1)***

attempting to solve for n ***(M1)***

$n \geq 31.68\ldots$

so the least value of n is 32 ***A1***

[6 marks]

Total [9 marks]

6. (a) $3-\frac{t}{2}=0 \Rightarrow t=6\,(\text{s})$ ***(M1)A1***

[2 marks]

Note: Award ***A0*** if either $t=-0.236$ or $t=4.24$ or both are stated with $t=6$.

(b) let d be the distance travelled before coming to rest

$d=\int_0^4 5-(t-2)^2\,\text{d}t+\int_4^6 3-\frac{t}{2}\text{d}t$ ***(M1)(A1)***

Note: Award ***M1*** for two correct integrals even if the integration limits are incorrect. The second integral can be specified as the area of a triangle.

$d=\frac{47}{3}\,(=15.7)\,(\text{m})$ ***(A1)***

attempting to solve $\int_6^T\left(\frac{t}{2}-3\right)\text{d}t=\frac{47}{3}$ (or equivalent) for T ***M1***

$T=13.9\,(\text{s})$ ***A1***

[5 marks]

Total [7 marks]

7. (a) each triangle has area $\frac{1}{8}x^2\sin\frac{2\pi}{n}$ $\left(\text{use of } \frac{1}{2}ab\sin C\right)$ ***(M1)***

there are n triangles so $A=\frac{1}{8}nx^2\sin\frac{2\pi}{n}$ ***A1***

$C=\frac{4\left(\frac{1}{8}nx^2\sin\frac{2\pi}{n}\right)}{\pi x^2}$ ***A1***

so $C=\frac{n}{2\pi}\sin\frac{2\pi}{n}$ ***AG***

[3 marks]

continued…

Question 7 continued

(b) attempting to find the least value of *n* such that $\frac{n}{2\pi}\sin\frac{2\pi}{n} > 0.99$ ***(M1)***

$n = 26$ ***A1***

attempting to find the least value of *n* such that $\dfrac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)} > 0.99$ ***(M1)***

$n = 21$ and so a regular polygon with 21 sides ***A1***

Note: Award ***(M0)A0(M1)A1*** if $\frac{n}{2\pi}\sin\frac{2\pi}{n} > 0.99$ is not considered and $\dfrac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)} > 0.99$ is correctly considered.
Award ***(M1)A1(M0)A0*** for $n = 26$.

[4 marks]

(c) **EITHER**

for even and odd values of *n*, the value of *C* seems to increase towards the limiting value of the circle $(C = 1)$ *ie* as *n* increases, the polygonal regions get closer and closer to the enclosing circular region ***R1***

OR

the differences between the odd and even values of *n* illustrate that this measure of compactness is not a good one ***R1***

[1 mark]

Total [8 marks]

Section B

8. (a) use of $A = \frac{1}{2} qr \sin\theta$ to obtain $A = \frac{1}{2}(x+2)(5-x)^2 \sin 30^\circ$ ***M1***

$= \frac{1}{4}(x + 2)(25 - 10x + x^2)$ ***A1***

$A = \frac{1}{4}(x^3 - 8x^2 + 5x + 50)$ ***AG***

[2 marks]

(b) (i) $\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{1}{4}\left(3x^2 - 16x + 5\right)\left(= \frac{1}{4}(3x - 1)(x - 5)\right)$ ***A1***

(ii) **METHOD 1**

EITHER

$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{1}{4}\left(3\left(\frac{1}{3}\right)^2 - 16\left(\frac{1}{3}\right) + 5\right) = 0$ ***M1A1***

OR

$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{1}{4}\left(3\left(\frac{1}{3}\right) - 1\right)\left(\left(\frac{1}{3}\right) - 5\right) = 0$ ***M1A1***

THEN

so $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ when $x = \frac{1}{3}$ ***AG***

METHOD 2

solving $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ for x ***M1***

$-2 < x < 5 \Rightarrow x = \frac{1}{3}$ ***A1***

so $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ when $x = \frac{1}{3}$ ***AG***

METHOD 3

a correct graph of $\frac{\mathrm{d}A}{\mathrm{d}x}$ versus x ***M1***

the graph clearly showing that $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ when $x = \frac{1}{3}$ ***A1***

so $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ when $x = \frac{1}{3}$ ***AG***

[3 marks]

continued…

Question 8 continued

(c) (i) $\dfrac{\mathrm{d}^2A}{\mathrm{d}x^2}=\dfrac{1}{2}(3x-8)$ ***A1***

for $x=\dfrac{1}{3}$, $\dfrac{\mathrm{d}^2A}{\mathrm{d}x^2}=-3.5\,(<0)$ ***R1***

so $x=\dfrac{1}{3}$ gives the maximum area of triangle PQR ***AG***

(ii) $A_{\max}=\dfrac{343}{27}\,(=12.7)\,(\text{cm}^2)$ ***A1***

(iii) $\text{PQ}=\dfrac{7}{3}\,(\text{cm})$ and $\text{PR}=\left(\dfrac{14}{3}\right)^2\,(\text{cm})$ ***(A1)***

$\text{QR}^2=\left(\dfrac{7}{3}\right)^2+\left(\dfrac{14}{3}\right)^4-2\left(\dfrac{7}{3}\right)\left(\dfrac{14}{3}\right)^2\cos 30^\circ$ ***(M1)(A1)***

$=391.702\ldots$

$\text{QR}=19.8\,(\text{cm})$ ***A1***

[7 marks]

Total [12 marks]

9.

(a) (i) $\text{P}(X=0)=0.549\left(=\mathrm{e}^{-0.6}\right)$ ***A1***

(ii) $\text{P}(X\geq 3)=1-\text{P}(X\leq 2)$ ***(M1)***

$\text{P}(X\geq 3)=0.0231$ ***A1***

[3 marks]

(b) **EITHER**

using $Y\sim\text{Po}(3)$ ***(M1)***

OR

using $(0.549)^5$ ***(M1)***

THEN

$\text{P}(Y=0)=0.0498\left(=\mathrm{e}^{-3}\right)$ ***A1***

[2 marks]

continued…

Question 9 continued

(c) $\mathrm{P}(X = 0)$ (most likely number of complaints received is zero) ***A1***

EITHER

calculating $\mathrm{P}(X = 0) = 0.549$ and $\mathrm{P}(X = 1) = 0.329$ ***M1A1***

OR

sketching an appropriate (discrete) graph of $\mathrm{P}(X = x)$ against x ***M1A1***

OR

finding $\mathrm{P}(X = 0) = e^{-0.6}$ and stating that $\mathrm{P}(X = 0) > 0.5$ ***M1A1***

OR

using $\mathrm{P}(X = x) = \mathrm{P}(X = x-1) \times \dfrac{\mu}{x}$ where $\mu < 1$ ***M1A1***

[3 marks]

(d) $\mathrm{P}(X = 0) = 0.8 \Rightarrow e^{-\lambda} = 0.8$ ***(A1)***

$\lambda = 0.223\left(= \ln\dfrac{5}{4}, = -\ln\dfrac{4}{5}\right)$ ***A1***

[2 marks]

Total [10 marks]

10. (a) attempting to use $V = \pi\int_a^b x^2 \mathrm{d}y$ ***(M1)***

attempting to express x^2 in terms of y *ie* $x^2 = 4(y+16)$ ***(M1)***

for $y = h$, $V = 4\pi\int_0^h y + 16\,\mathrm{d}y$ ***A1***

$V = 4\pi\left(\dfrac{h^2}{2} + 16h\right)$ ***AG***

[3 marks]

continued…

Question 10 continued

(b) (i) **METHOD 1**

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$ ***(M1)***

$$\frac{dV}{dh} = 4\pi(h+16)$$ ***(A1)***

$$\frac{dh}{dt} = \frac{1}{4\pi(h+16)} \times \frac{-250\sqrt{h}}{\pi(h+16)}$$ ***M1A1***

Note: Award ***M1*** for substitution into $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$.

$$\frac{dh}{dt} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$$ ***AG***

METHOD 2

$$\frac{dV}{dt} = 4\pi(h+16)\frac{dh}{dt}$$ (implicit differentiation) ***(M1)***

$$\frac{-250\sqrt{h}}{\pi(h+16)} = 4\pi(h+16)\frac{dh}{dt}$$ (or equivalent) ***A1***

$$\frac{dh}{dt} = \frac{1}{4\pi(h+16)} \times \frac{-250\sqrt{h}}{\pi(h+16)}$$ ***M1A1***

$$\frac{dh}{dt} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$$ ***AG***

(ii) $$\frac{dt}{dh} = -\frac{4\pi^2(h+16)^2}{250\sqrt{h}}$$ ***A1***

$$t = \int -\frac{4\pi^2(h+16)^2}{250\sqrt{h}}\, dh$$ ***(M1)***

$$t = \int -\frac{4\pi^2(h^2+32h+256)}{250\sqrt{h}}\, dh$$ ***A1***

$$t = \frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)dh$$ ***AG***

continued…

Question 10 continued

(iii) **METHOD 1**

$$t=\int_{48}^{0}\left(h^{\frac{3}{2}}+32h^{\frac{1}{2}}+256h^{-\frac{1}{2}}\right)\mathrm{d}h$$ ***(M1)(A1)***

$t=2688.756\ldots\ (\mathrm{s})$ ***(A1)***

45 minutes (correct to the nearest minute) ***A1***

METHOD 2

$$t=\frac{-4\pi^2}{250}\left(\frac{2}{5}h^{\frac{5}{2}}+\frac{64}{3}h^{\frac{3}{2}}+512h^{\frac{1}{2}}\right)+c$$ ***A1***

$$t=0, h=48\Rightarrow c=2688.756\ldots\left(c=\frac{4\pi^2}{250}\left(\frac{2}{5}\times 48^{\frac{5}{2}}+\frac{64}{3}\times 48^{\frac{3}{2}}+512\times 48^{\frac{1}{2}}\right)\right)$$ ***(M1)***

$$h=0,\ t=2688.756\ldots\left(t=\frac{4\pi^2}{250}\left(\frac{2}{5}\times 48^{\frac{5}{2}}+\frac{64}{3}\times 48^{\frac{3}{2}}+512\times 48^{\frac{1}{2}}\right)\right)(\mathrm{s})$$ ***(A1)***

45 minutes (correct to the nearest minute) ***A1***

[11 marks]

(c) **EITHER**

the depth stabilizes when $\frac{\mathrm{d}V}{\mathrm{d}t}=0$ *ie* $8.5-\frac{250\sqrt{h}}{\pi(h+16)}=0$ ***R1***

attempting to solve $8.5-\frac{250\sqrt{h}}{\pi(h+16)}=0$ for h ***(M1)***

OR

the depth stabilizes when $\frac{\mathrm{d}h}{\mathrm{d}t}=0$ *ie* $\frac{1}{4\pi(h+16)}\left(8.5-\frac{250\sqrt{h}}{\pi(h+16)}\right)=0$ ***R1***

attempting to solve $\frac{1}{4\pi(h+16)}\left(8.5-\frac{250\sqrt{h}}{\pi(h+16)}\right)=0$ for h ***(M1)***

THEN

$h=5.06\ (\mathrm{cm})$ ***A1***

[3 marks]

Total [17 marks]

11. (a) squaring both equations ***M1***

$9\sin^2 B + 24\sin B\cos C + 16\cos^2 C = 36$ ***(A1)***

$9\cos^2 B + 24\cos B\sin C + 16\sin^2 C = 1$ ***(A1)***

adding the equations and using $\cos^2\theta + \sin^2\theta = 1$ to

obtain $9 + 24(\sin B\cos C + \cos B\sin C) + 16 = 37$ ***M1***

$24(\sin B\cos C + \cos B\sin C) = 12$ ***A1***

$24\sin(B + C) = 12$ ***(A1)***

$\sin(B + C) = \frac{1}{2}$ ***AG***

[6 marks]

(b) $\sin A = \sin\left(180^\circ - (B + C)\right)$ so $\sin A = \sin(B + C)$ ***R1***

$\sin(B + C) = \frac{1}{2} \Rightarrow \sin A = \frac{1}{2}$ ***A1***

$\Rightarrow A = 30^\circ$ or $A = 150^\circ$ ***A1***

Note: Award ***R1A1A1*** for obtaining $B + C = 30^\circ$ or $B + C = 150^\circ$.

if $A = 150^\circ$, then $B < 30^\circ$ ***R1***

for example, $3\sin B + 4\cos C < \left(\frac{3}{2} + 4\right) < 6$, *ie* a contradiction ***R1***

only one possible value $(A = 30^\circ)$ ***AG***

[5 marks]

Total [11 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 3 – discrete mathematics

SPECIMEN (adapted from November 2014)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[50 marks]**.

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Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 8]

Let $f(n) = n^5 - n$, $n \in \mathbb{Z}^+$.

(a) Find the values of $f(3)$ and $f(4)$. [1]

(b) Use the Euclidean algorithm to find $\gcd\left(f(3), f(4)\right)$. [2]

(c) Use Fermat's Little Theorem to explain why $f(n)$ is always exactly divisible by 5. [1]

(d) By factorizing $f(n)$ explain why it is always exactly divisible by 6. [4]

2. [Maximum mark: 8]

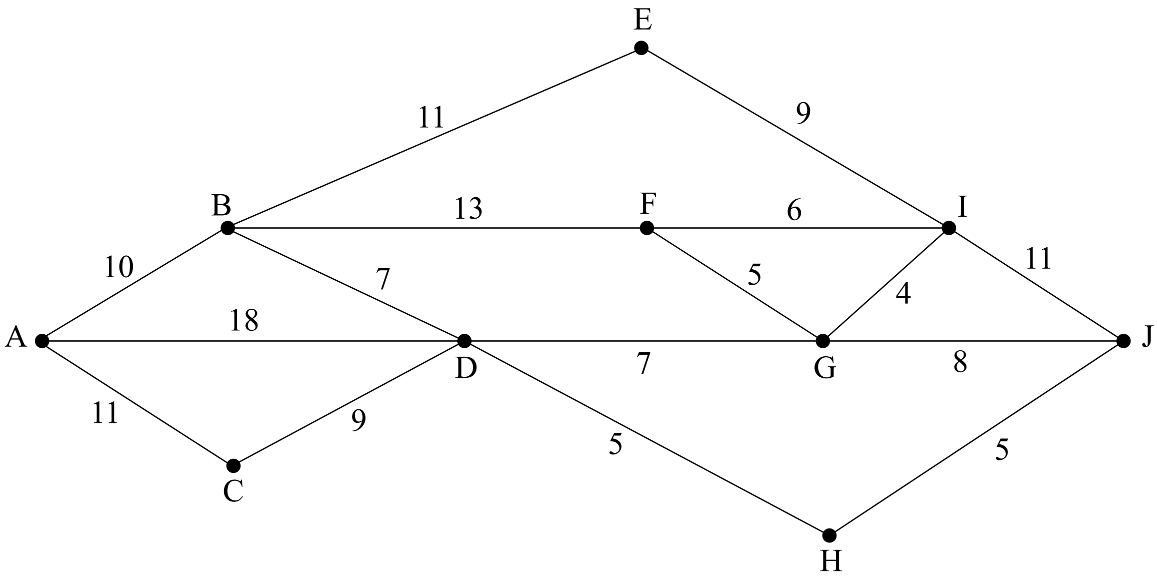
(a) Use the pigeon-hole principle to prove that for any simple graph that has two or more vertices and in which every vertex is connected to at least one other vertex, there must be at least two vertices with the same degree. [4]

Seventeen people attend a meeting.

(b) Each person shakes hands with at least one other person and no-one shakes hands with the same person more than once. Use the result from part (a) to show that there must be at least two people who shake hands with the same number of people. [4]

3. [Maximum mark: 13]

The following graph represents the cost in dollars of travelling by bus between 10 towns in a particular province.



(a) Use Dijkstra's algorithm to find the cheapest route between A and J, and state its cost. [7]

For the remainder of the question you may find the cheapest route between any two towns by inspection.

It is given that the total cost of travelling on all the roads without repeating any is $139. A tourist decides to go over all the roads at least once, starting and finishing at town A.

(b) Find the lowest possible cost of his journey, stating clearly which roads need to be travelled more than once. You must fully justify your answer. [6]

4. [Maximum mark: 10]

(a) Solve, by any method, the following system of linear congruences

$$x \equiv 9 (\text{mod}\, 11)$$
$$x \equiv 1 (\text{mod}\, 5).$$ [3]

(b) Find the remainder when 41^{82} is divided by 11. [4]

(c) Using your answers to parts (a) and (b) find the remainder when 41^{82} is divided by 55. [3]

5. [Maximum mark: 11]

Andy and Roger are playing tennis with the rule that if one of them wins a game when serving then he carries on serving, and if he loses then the other player takes over the serve.

The probability Andy wins a game when serving is $\frac{1}{2}$ and the probability he wins a game when not serving is $\frac{1}{4}$. Andy serves in the first game. Let u_n denote the probability that Andy wins the nth game.

(a) State the value of u_1. [1]

(b) Show that u_n satisfies the recurrence relation

$$u_n = \frac{1}{4}u_{n-1} + \frac{1}{4}.$$ [4]

(c) Solve this recurrence relation to find the probability that Andy wins the nth game. [6]

Markscheme

Specimen (adapted from November 2014)

Discrete mathematics

Higher level

Paper 3

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2016**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.
  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685\ldots$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working*.

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER… OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

Calculator notation The mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) $240,\ 1020$ ***A1***

Note: Award ***A2*** for three correct answers, ***A1*** for two correct answers.

[1 mark]

(b) $1020 = 240 \times 4 + 60$ ***(M1)***

$240 = 60 \times 4$

$\gcd(1020,\ 240) = 60$ ***A1***

[2 marks]

Note: Must be done by Euclid's algorithm.

(c) by Fermat's little theorem with $p = 5$

$n^5 \equiv n \pmod 5$ ***A1***

so 5 divides $f(n)$

[1 mark]

(d) $f(n) = n(n^2 - 1)(n^2 + 1) = n(n-1)(n+1)(n^2+1)$ ***(A1)A1***

$n-1,\ n,\ n+1$ are consecutive integers and so contain a multiple of 2 and 3 ***R1R1***

Note: Award ***R1*** for justification of 2 and ***R1*** for justification of 3.

and therefore $f(n)$ is a multiple of 6 ***AG***

[4 marks]

Total [8 marks]

2. (a) let there be v vertices in the graph; because the graph is simple the degree of each vertex is $\leq v-1$ ***A1***

the degree of each vertex is ≥ 1 ***A1***

there are therefore $v-1$ possible values for the degree of each vertex ***A1***

given there are v vertices by the pigeon-hole principle there must be at least two with the same degree ***R1***

[4 marks]

(b) consider a graph in which the people at the meeting are represented by the vertices and two vertices are connected if the two people shake hands ***M1***

the graph is simple as no-one shakes hands with the same person more than once (nor can someone shake hands with themselves) ***A1***

every vertex is connected to at least one other vertex as everyone shakes at least one hand ***A1***

the degree of each vertex is the number of handshakes so by the proof above there must be at least two who shake the same number of hands ***R1***

[4 marks]

Total [8 marks]

3. (a) ***M1A1A1A1***

	A	B	C	D	E	F	G	H	I	J
A	$\boxed{0}$	10	11	18						
B		$\boxed{10}$		17	21	23				
C			$\boxed{11}$							
D				$\boxed{17}$			24	22		
E					$\boxed{21}$				30	
H								$\boxed{22}$		27
F						$\boxed{23}$			29	
G							$\boxed{24}$		28	
J										$\boxed{27}$

(***M1*** for an attempt at Dijkstra's)
(***A1*** for value of $D = 17$)
(***A1*** for value of $H = 22$)
(***A1*** for value of $G = 24$)

route is ABDHJ ***(M1)A1***
cost is \$27 ***A1***

Note: Accept other layouts.

[7 marks]

(b) there are 4 odd vertices A, D, F and J ***A1***
these can be joined up in 3 ways with the following extra costs
AD and FJ $17+13=30$
AF and DJ $23+10=33$
AJ and DF $27+12=39$ ***M1A1A1***

Notes: Award ***M1*** for an attempt to find different routes.
Award ***A1A1*** for correct values for all three costs ***A1*** for one correct.

need to repeat AB, BD, FG and GJ ***A1***
total cost is $139+30=\$169$ ***A1***

[6 marks]

Total [13 marks]

4. (a) **METHOD 1**

listing $9, 20, 31, \ldots$ and $1, 6, 11, 16, 21, 26, 31, \ldots$ ***M1***

one solution is 31 ***(A1)***

by the Chinese remainder theorem the full solution is

$x \equiv 31 (\bmod 55)$ ***A1 N2***

METHOD 2

$x \equiv 9(\bmod 11) \Rightarrow x = 9 + 11t$ ***M1***

$\Rightarrow 9 + 11t \equiv 1(\bmod 5)$

$\Rightarrow t \equiv 2(\bmod 5)$ ***A1***

$\Rightarrow t = 2 + 5s$

$\Rightarrow x = 9 + 11(2 + 5s)$

$\Rightarrow x = 31 + 55s \left(\Rightarrow x \equiv 31(\bmod 55)\right)$ ***A1***

Note: Accept other methods *eg* formula, Diophantine equation.

Note: Accept other equivalent answers *eg* –79(mod55).

[3 marks]

(b) $41^{82} \equiv 8^{82} (\bmod 11)$

by Fermat's little theorem $8^{10} \equiv 1(\bmod 11)$ (or $41^{10} \equiv 1(\bmod 11)$) ***M1***

$8^{82} \equiv 8^{2} (\bmod 11)$ ***M1***

$\equiv 9 (\bmod 11)$ ***(A1)***

remainder is 9 ***A1***

[4 marks]

Note: Accept simplifications done without Fermat.

(c) $41^{82} \equiv 1^{82} \equiv 1(\bmod 5)$ ***A1***

so 41^{82} has a remainder 1 when divided by 5 and a remainder 9 when divided by 11 ***R1***

hence by part (a) the remainder is 31 ***A1***

[3 marks]

Total [10 marks]

5. (a) $\frac{1}{2}$ ***A1***

[1 mark]

(b) Andy could win the nth game by winning the $n-1$th and then winning the nth game or by losing the $n-1$th and then winning the nth ***(M1)***

$u_n = \frac{1}{2}u_{n-1} + \frac{1}{4}(1 - u_{n-1})$ ***A1A1M1***

Note: Award ***A1*** for each term and ***M1*** for addition of two probabilities.

$u_n = \frac{1}{4}u_{n-1} + \frac{1}{4}$ ***AG***

[4 marks]

(c) general solution is $u_n = A\left(\frac{1}{4}\right)^n + p(n)$ ***(M1)***

for a particular solution try $p(n) = b$ ***(M1)***

$b = \frac{1}{4}b + \frac{1}{4}$ ***(A1)***

$b = \frac{1}{3}$

hence $u_n = A\left(\frac{1}{4}\right)^n + \frac{1}{3}$ ***(A1)***

using $u_1 = \frac{1}{2}$ ***M1***

$\frac{1}{2} = A\left(\frac{1}{4}\right) + \frac{1}{3} \Rightarrow A = \frac{2}{3}$

hence $u_n = \frac{2}{3}\left(\frac{1}{4}\right)^n + \frac{1}{3}$ ***A1***

Note: Accept other valid methods.

[6 marks]

Total [11 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 3 – calculus

SPECIMEN (adapted from November 2014)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[50 marks]**.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 14]

(a) Use the integral test to determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.5}} .$$ [3]

(b) Let $S = \sum_{n=1}^{\infty} \frac{(x+1)^n}{2^n \times n^{0.5}}$.

(i) Use the ratio test to show that S is convergent for $-3 < x < 1$.

(ii) Hence find the interval of convergence for S. [11]

2. [Maximum mark: 13]

(a) Use an integrating factor to show that the general solution for $\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{x}{t} = -\frac{2}{t}$, $t > 0$ is $x = 2 + ct$, where c is a constant. [4]

The weight in kilograms of a dog, t weeks after being bought from a pet shop, can be modelled by the following function

$$w(t) = \begin{cases} 2 + ct & 0 \le t \le 5 \\ 16 - \dfrac{35}{t} & t > 5 \end{cases} .$$

(b) Given that $w(t)$ is continuous, find the value of c . [2]

(c) Write down an upper bound for the weight of the dog. [1]

(d) Prove from first principles that $w(t)$ is differentiable at $t = 5$. [6]

3. [Maximum mark: 10]

Consider the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ where $f(x, y) = y - 2x$.

(a) Sketch, on one diagram, the four isoclines corresponding to $f(x, y) = k$ where k takes the values -1, -0.5, 0 and 1. Indicate clearly where each isocline crosses the y-axis. [2]

A curve, C, passes through the point $(0, 1)$ and satisfies the differential equation above.

(b) Sketch C on your diagram. [3]

(c) State a particular relationship between the isocline $f(x, y) = -0.5$ and the curve C, at their point of intersection. [1]

(d) Use Euler's method with a step interval of 0.1 to find an approximate value for y on C, when $x = 0.5$. [4]

4. [Maximum mark: 13]

In this question you may assume that $\arctan x$ is continuous and differentiable for $x \in \mathbb{R}$.

(a) Consider the infinite geometric series

$$1 - x^2 + x^4 - x^6 + \ldots \qquad |x| < 1.$$

Show that the sum of the series is $\dfrac{1}{1+x^2}$. [1]

(b) Hence show that an expansion of $\arctan x$ is $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$ [4]

(c) f is a continuous function defined on $[a, b]$ and differentiable on $]a, b[$ with $f'(x) > 0$ on $]a, b[$.

Use the mean value theorem to prove that for any $x, y \in [a, b]$, if $y > x$ then $f(y) > f(x)$. [4]

(d) (i) Given $g(x) = x - \arctan x$, prove that $g'(x) > 0$, for $x > 0$.

(ii) Use the result from part (c) to prove that $\arctan x < x$, for $x > 0$. [4]

Markscheme

Specimen (adapted from November 2014)

Calculus

Higher level

Paper 3

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2016**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.
  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685\ldots$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 ***N*** marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER… OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

Calculator notation The mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) $\int_1^{\infty} x^{-0.5}\,\mathrm{d}x$ ***M1***

$= \lim_{H\to\infty}\left[2x^{0.5}\right]_1^H$ ***A1***

Note: Accept $\left[2x^{0.5}\right]_1^{\infty}$.

this is not finite so series is divergent ***R1***

Notes: Accept equivalent eg $\to\infty$, or "limit does not exist". If lower limit is not equal to 1 award ***M0A0***, but the ***R1*** can still be awarded if the final reasoning is correct.

[3 marks]

(b) (i) applying the ratio test ***M1***

$\lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{2^{n+1}(n+1)^{0.5}}\times\frac{2^n n^{0.5}}{(x+1)^n}\right|$ ***A1***

$\lim_{n\to\infty}\left|\frac{(x+1)n^{0.5}}{2(n+1)^{0.5}}\right| = \left|\frac{(x+1)}{2}\right|$ ***A1***

Note: Do not penalize the absence of limits and modulus signs.

converges if $\left|\frac{x+1}{2}\right| < 1 \Rightarrow -1 < \frac{(x+1)}{2} < 1$ ***M1***

$\Rightarrow -3 < x < 1$ ***A1***

Note: Accept $-2 < x+1 < 2$.

(ii) considering end points ***M1***

when $x = -3$, series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{0.5}}$ ***A1***

$\frac{1}{n^{0.5}}$ is a decreasing sequence with limit zero, ***R1***

so series converges by alternating series test ***R1***

when $x = 1$, series is $\sum_{n=1}^{\infty}\frac{1}{n^{0.5}}$ which diverges by part (a) or p-series ***A1***

Note: This ***A1*** is for both the reasoning and the statement it diverges.

interval of convergence is $-3 \le x < 1$ ***A1***

[11 marks]

Total [14 marks]

2. (a) integrating factor $e^{\int -\frac{1}{t}\mathrm{d}t} = e^{-\ln t}\left(= \frac{1}{t}\right)$ ***M1A1***

$$\frac{x}{t} = \int -\frac{2}{t^2}\mathrm{d}t = \frac{2}{t} + c$$ ***A1A1***

Note: Award ***A1*** for $\frac{x}{t}$ and ***A1*** for $\frac{2}{t}+c$.

$x = 2 + ct$ ***AG***

[4 marks]

(b) given continuity at $x = 5$

$5c + 2 = 16 - \frac{35}{5} \Rightarrow c = \frac{7}{5}$ ***M1A1***

[2 marks]

(c) any value ≥ 16 ***A1***

Note: Accept values less than 16 if fully justified by reference to the maximum age for a dog.

[1 mark]

(d) $\lim\limits_{h\to 0-}\left(\dfrac{\frac{7}{5}(5+h)+2-\frac{7}{5}(5)-2}{h}\right) = \frac{7}{5}$ ***M1A1***

$\lim\limits_{h\to 0+}\left(\dfrac{16-\frac{35}{5+h}-16+\frac{35}{5}}{h}\right)\left(=\lim\limits_{h\to 0+}\left(\dfrac{\frac{-35}{5+h}+7}{h}\right)\right)$ ***M1***

$= \lim\limits_{h\to 0+}\left(\dfrac{\frac{-35+35+7h}{(5+h)}}{h}\right) = \lim\limits_{h\to 0+}\left(\dfrac{7}{5+h}\right) = \frac{7}{5}$ ***M1A1***

both limits equal so differentiable at $t = 5$ ***R1AG***

Notes: The limits $t \to 5$ could also be used.

For each value of $\frac{7}{5}$ obtained by standard differentiation award ***A1***.

To gain the other ***[4 marks]*** a rigorous explanation must be given on how you can get from the left and right hand derivatives to the derivative.

Notes: If the candidate works with *t* and then substitutes $t = 5$ at the end award as follows.

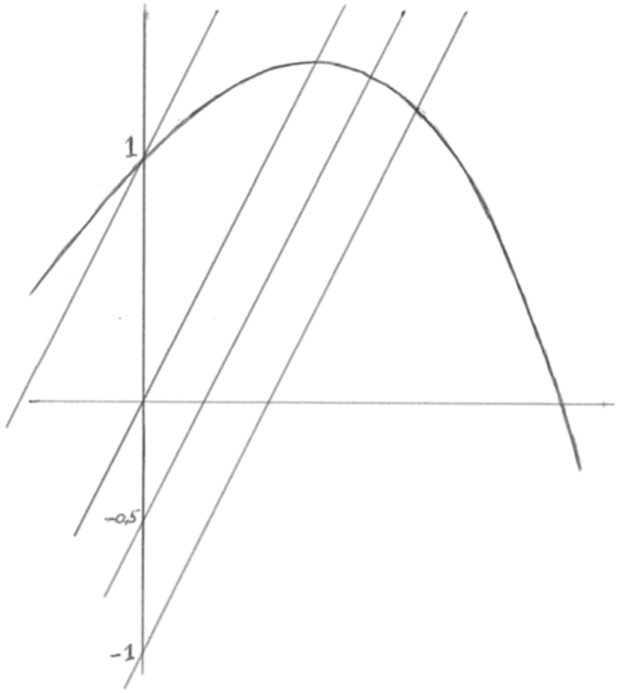
First ***M1*** for using formula with *t* in the linear case, ***A1*** for $\frac{7}{5}$.

Award next 2 method marks even if $t = 5$ not substituted, ***A1*** for $\frac{7}{5}$.

[6 marks]

Total [13 marks]

3. (a) and (b)



(a) ***A1*** for 4 parallel straight lines with a positive gradient ***A1***
A1 for correct *y* intercepts ***A1***

[2 marks]

(b) ***A1*** for passing through $(0,1)$ with positive gradient less than 2
A1 for stationary point on $y = 2x$
A1 for negative gradient on both of the other 2 isoclines ***A1A1A1***

[3 marks]

(c) the isocline is perpendicular to C ***R1***

[1 mark]

(d) $y_{n+1} = y_n + 0.1(y_n - 2x_n)(= 1.1y_n - 0.2x_n)$ ***(M1)(A1)***

Note: Also award ***M1A1*** if no formula seen but y_2 is correct.

$y_0 = 1,\ y_1 = 1.1,\ y_2 = 1.19,\ y_3 = 1.269,\ y_4 = 1.3359$ ***(M1)***

$y_5 = 1.39$ to 3 sf ***A1***

Note: ***M1*** is for repeated use of their formula, with steps of 0.1.

Note: Accept 1.39 or 1.4 only.

[4 marks]

Total [10 marks]

4. (a) $r = -x^2$, $S = \dfrac{1}{1+x^2}$ ***A1AG***

[1 mark]

(b) $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$

EITHER

$\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x = \int 1 - x^2 + x^4 - x^6 + \ldots\,\mathrm{d}x$ ***M1***

$\arctan x = c + x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$ ***A1***

Note: Do not penalize the absence of c at this stage.

when $x = 0$ we have $\arctan 0 = c$ hence $c = 0$ ***M1A1***

OR

$\displaystyle\int_0^x \frac{1}{1+t^2}\,\mathrm{d}t = \int_0^x 1 - t^2 + t^4 - t^6 + \ldots\,\mathrm{d}t$ ***M1A1A1***

Notes: Allow x as the variable as well as the limit.
M1 for knowing to integrate, ***A1*** for each of the limits.

$\left[\arctan t\right]_0^x = \left[t - \dfrac{t^3}{3} + \dfrac{t^5}{5} - \dfrac{t^7}{7} + \ldots\right]_0^x$ ***A1***

hence $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$ ***AG***

[4 marks]

(c) applying the MVT to the function f on the interval $[x, y]$ ***M1***

$\dfrac{f(y) - f(x)}{y - x} = f'(c)$ (for some $c \in \left]x, y\right[$) ***A1***

$\dfrac{f(y) - f(x)}{y - x} > 0$ (as $f'(c) > 0$) ***R1***

$f(y) - f(x) > 0$ as $y > x$ ***R1***

$\Rightarrow f(y) > f(x)$ ***AG***

[4 marks]

Note: If they use x rather than c they should be awarded ***M1A0R0***, but could get the next ***R1***.

continued…

Question 4 continued

(d) (i) $g(x) = x - \arctan x \Rightarrow g'(x) = 1 - \dfrac{1}{1 + x^2}$ ***A1***

this is greater than zero because $\dfrac{1}{1+x^2} < 1$ ***R1***

so $g'(x) > 0$ ***AG***

(ii) (g is a continuous function defined on $[0, b]$ and differentiable on $]0, b[$ with $g'(x) > 0$ on $]0, b[$ for all $b \in \mathbb{R}$)

(if $x \in [0, b]$ then) from part (c) $g(x) > g(0)$ ***M1***

$x - \arctan x > 0 \Rightarrow \arctan x < x$ ***M1***

(as b can take any positive value it is true for all $x > 0$) ***AG***

[4 marks]

Total [13 marks]

Mathematics
Higher level
Paper 3 – sets, relations and groups

SPECIMEN (adapted from November 2014)

1 hour

Instructions to candidates

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- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[50 marks]**.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 12]

A group with the binary operation of multiplication modulo 15 is shown in the following Cayley table.

$\times_{15}$	**1**	**2**	**4**	**7**	**8**	**11**	**13**	**14**
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	a	b	c
13	13	11	7	1	14	d	e	f
14	14	13	11	8	7	g	h	i

(a) Find the values represented by each of the letters in the table. [3]

(b) Find the order of each of the elements of the group. [3]

(c) Write down the three sets that form subgroups of order 2. [2]

(d) Find the three sets that form subgroups of order 4. [4]

2. [Maximum mark: 8]

Define $f:\mathbb{R}\setminus\{0.5\}\to\mathbb{R}$ by $f(x)=\dfrac{4x+1}{2x-1}$.

(a) Prove that f is an injection. [4]

(b) Prove that f is not a surjection. [4]

3. [Maximum mark: 9]

Consider the set A consisting of all the permutations of the integers $1, 2, 3, 4, 5$.

(a) Two members of A are given by $p=(1\ 2\ 5)$ and $q=(1\ 3)(2\ 5)$.
Find the single permutation which is equivalent to $q\circ p$. [3]

(b) State a permutation belonging to A of order 6. [2]

(c) Let $P=$ {all permutations in A where exactly two integers change position},

and $Q=$ {all permutations in A where the integer 1 changes position}.

(i) List all the elements in $P\cap Q$.

(ii) Find $n(P\cap Q')$. [4]

4. [Maximum mark: 10]

The group $\{G, *\}$ has identity e_G and the group $\{H, \circ\}$ has identity e_H. A homomorphism f is such that $f: G \to H$. It is given that $f(e_G) = e_H$.

(a) Prove that for all $a \in G, f(a^{-1}) = (f(a))^{-1}$. [4]

Let $\{H, \circ\}$ be the cyclic group of order seven, and let p be a generator.
Let $x \in G$ such that $f(x) = p^2$.

(b) Find $f(x^{-1})$. [2]

(c) Given that $f(x * y) = p$, find $f(y)$. [4]

5. [Maximum mark: 11]

$\{G, *\}$ is a group with identity element e. Let $a, b \in G$.

(a) Verify that the inverse of $a * b^{-1}$ is equal to $b * a^{-1}$. [3]

Let $\{H, *\}$ be a subgroup of $\{G, *\}$. Let R be a relation defined on G by

$$aRb \Leftrightarrow a * b^{-1} \in H.$$

(b) Prove that R is an equivalence relation, indicating clearly whenever you are using one of the four properties required of a group. [8]

Markscheme

Specimen (adapted from November 2014)

Sets, relations and groups

Higher level

Paper 3

9 pages

Instructions to Examiners

Abbreviations

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R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2016**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
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  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685\ldots$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

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*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

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$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

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If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

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A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

Calculator notation The mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) $a=1$, $b=8$, $c=4$,
$d=8$, $e=4$, $f=2$,
$g=4$, $h=2$, $i=1$ ***A3***

Note: Award ***A3*** for 9 correct answers, ***A2*** for 6 or more, and ***A1*** for 3 or more.

[3 marks]

(b)

Elements	**Order**
1	1
4, 11, 14	2
2, 7, 8, 13	4

A3

Note: Award ***A3*** for 8 correct answers, ***A2*** for 6 or more, and ***A1*** for 4 or more.

[3 marks]

(c) $\{1, 4\}, \{1, 11\}, \{1, 14\}$ ***A1A1***

Note: Award ***A1*** for 1 correct answer and ***A2*** for all 3 (and no extras).

[2 marks]

(d) $\{1, 2, 4, 8\}$, $\{1, 4, 7, 13\}$, ***A1A1***
$\{1, 4, 11, 14\}$ ***A2***

[4 marks]

Total [12 marks]

2. (a) **METHOD 1**

$f(x) = f(y) \Rightarrow \dfrac{4x+1}{2x-1} = \dfrac{4y+1}{2y-1}$ ***M1A1***

for attempting to cross multiply and simplify ***M1***

$(4x+1)(2y-1) = (2x-1)(4y+1)$

$\Rightarrow 8xy + 2y - 4x - 1 = 8xy + 2x - 4y - 1 \Rightarrow 6y = 6x$

$\Rightarrow x = y$ ***A1***

hence an injection ***AG***

[4 marks]

METHOD 2

$f'(x) = \dfrac{4(2x-1) - 2(4x+1)}{(2x-1)^2} = \dfrac{-6}{(2x-1)^2}$ ***M1A1***

< 0 (for all $x \neq 0.5$) ***R1***

therefore the function is decreasing on either side of the discontinuity and

$f(x) < 2$ for $x < 0.5$ and $x > 2$ for $f(x) > 0.5$ ***R1***

hence an injection ***AG***

Note:	If a correct graph of the function is shown, and the candidate states this is decreasing in each part (or horizontal line test) and hence an injection, award ***M1A1R1***.

[4 marks]

(b) **METHOD 1**

attempt to solve $y = \dfrac{4x+1}{2x-1}$ ***M1***

$y(2x-1) = 4x+1 \Rightarrow 2xy - y = 4x+1$ ***A1***

$2xy - 4x = 1 + y \Rightarrow x = \dfrac{1+y}{2y-4}$ ***A1***

no value for $y = 2$ ***R1***

hence not a surjection ***AG***

[4 marks]

METHOD 2

consider $y = 2$ ***A1***

attempt to solve $2 = \dfrac{4x+1}{2x-1}$ ***M1***

$4x - 2 = 4x + 1$ ***A1***

which has no solution ***R1***

hence not a surjection ***AG***

Note:	If a correct graph of the function is shown, and the candidate states that because there is a horizontal asymptote at $y = 2$ then the function is not a surjection, award ***M1R1***.

[4 marks]

Total [8 marks]

3. (a) $q \circ p = (1\ 3)\ (2\ 5)\ (1\ 2\ 5)$ ***(M1)***

$= (1\ 5\ 3)$ ***M1A1***

Note: ***M1*** for an answer consisting of disjoint cycles, ***A1*** for (1 5 3).

Note: Allow $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{pmatrix}$, allow $(1\ 5\ 3)(2)$. If done in the wrong order and obtained $(1\ 3\ 2)$, award ***A2***.

[3 marks]

(b) any permutation with 2 disjoint cycles one of length 2 and one of length 3 *eg* $(1\ 2)\ (3\ 4\ 5)$ ***M1A1***

Notes: Award ***M1A0*** for any permutation with 2 non-disjoint cycles one of length 2 and one of length 3. Accept non cycle notation.

[2 marks]

(c) (i) $(1,\ 2), (1,\ 3), (1,\ 4), (1,\ 5)$ ***M1A1***

(ii) $(2\ 3), (2\ 4), (2\ 5), (3\ 4), (3\ 5), (4\ 5)$ ***(M1)***

6 ***A1***

Note: Award ***M1*** for at least one correct cycle.

[4 marks]

Total [9 marks]

4. (a) $f(e_G) = e_H \Rightarrow f\left(a * a^{-1}\right) = e_H$ ***M1***

f is a homomorphism so $f\left(a * a^{-1}\right) = f(a) \circ f\left(a^{-1}\right) = e_H$ ***M1A1***

by definition $f(a) \circ \left(f(a)\right)^{-1} = e_H$ so $f\left(a^{-1}\right) = \left(f(a)\right)^{-1}$ (by the left-cancellation law) ***R1***

[4 marks]

(b) from (a) $f\left(x^{-1}\right) = \left(f(x)\right)^{-1}$

hence $f(x^{-1}) = (p^2)^{-1} = p^5$ ***M1A1***

[2 marks]

(c) $f(x * y) = f(x) \circ f(y)$ (homomorphism) ***(M1)***

$p^2 \circ f(y) = p$ ***A1***

$f(y) = p^5 \circ p$ ***(M1)***

$= p^6$ ***A1***

[4 marks]

Total [10 marks]

5. (a) **METHOD 1**

$\left(a*b^{-1}\right)*\left(b*a^{-1}\right)=a*b^{-1}*b*a^{-1}=a*e*a^{-1}=a*a^{-1}=e$ ***M1A1A1***

> **Notes:** ***M1*** for multiplying, ***A1*** for at least one of the next 3 expressions, ***A1*** for e.
> Allow $\left(b*a^{-1}\right)*\left(a*b^{-1}\right)=b*a^{-1}*a*b^{-1}=b*e*b^{-1}=b*b^{-1}=e$.

METHOD 2

$\left(a*b^{-1}\right)^{-1}=\left(b^{-1}\right)^{-1}*a^{-1}$ ***M1A1***

$=b*a^{-1}$ ***A1***

[3 marks]

(b) $a*a^{-1}=e\in H$ (as H is a subgroup) ***M1***

so aRa and hence R is reflexive

$aRb \Leftrightarrow a*b^{-1}\in H$. H is a subgroup so every element has an inverse in H so $\left(a*b^{-1}\right)\in H$ ***R1***

$\Leftrightarrow b*a^{-1}\in H \Leftrightarrow bRa$ ***M1***

so R is symmetric

$aRb,\ bRc \Leftrightarrow a*b^{-1}\in H,\ b*c^{-1}\in H$ ***M1***

as H is closed $\left(a*b^{-1}\right)*\left(b*c^{-1}\right)\in H$ ***R1***

and using associativity ***R1***

$\left(a*b^{-1}\right)*\left(b*c^{-1}\right)=a*\left(b^{-1}*b\right)*c^{-1}=a*c^{-1}\in H \Leftrightarrow aRc$ ***A1***

therefore R is transitive

R is reflexive, symmetric and transitive ***A1***

> **Note:** Can be said separately at the end of each part.

hence it is an equivalence relation ***AG***

[8 marks]

Total [11 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 3 – statistics and probability

SPECIMEN (adapted from November 2014)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[50 marks]**.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 7]

A random variable X has probability density function

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \le x < 1 \\ \frac{1}{4} & 1 \le x < 3 \\ 0 & x \ge 3 \end{cases}.$$

(a) Sketch the graph of $y = f(x)$. [1]

(b) Find the cumulative distribution function for X. [5]

(c) Find the upper quartile for X. [1]

2. [Maximum mark: 9]

Eric plays a game at a fairground in which he throws darts at a target. Each time he throws a dart, the probability of hitting the target is 0.2. He is allowed to throw as many darts as he likes, but it costs him $\$1$ a throw. If he hits the target a total of three times he wins $\$10$.

(a) Find the probability he has his third success of hitting the target on his sixth throw. [3]

(b) (i) Find the expected number of throws required for Eric to hit the target three times.

(ii) Write down his expected profit or loss if he plays until he wins the $\$10$. [3]

(c) If he has just $\$8$, find the probability he will lose all his money before he hits the target three times. [3]

3. [Maximum mark: 11]

(a) If X and Y are two random variables such that $\mathrm{E}(X) = \mu_X$ and $\mathrm{E}(Y) = \mu_Y$ then $\mathrm{Cov}(X, Y) = \mathrm{E}\big((X - \mu_X)(Y - \mu_Y)\big)$.

Prove that if X and Y are independent then $\mathrm{Cov}(X, Y) = 0$. [3]

(b) In a particular company, it is claimed that the distance travelled by employees to work is independent of their salary. To test this, 20 randomly selected employees are asked about the distance they travel to work and the size of their salaries. It is found that the product moment correlation coefficient, r, for the sample is -0.35.

You may assume that both salary and distance travelled to work follow normal distributions.

Perform a one-tailed test at the 5% significance level to test whether or not the distance travelled to work and the salaries of the employees are independent. [8]

4. [Maximum mark: 13]

If X is a random variable that follows a Poisson distribution with mean $\lambda > 0$ then the probability generating function of X is $G(t) = e^{\lambda(t-1)}$.

(a) (i) Prove that $\mathrm{E}(X) = \lambda$.

(ii) Prove that $\mathrm{Var}(X) = \lambda$. [6]

Y is a random variable, independent of X, that also follows a Poisson distribution with mean λ.

(b) If $S = 2X - Y$ find

(i) $\mathrm{E}(S)$;

(ii) $\mathrm{Var}(S)$. [3]

Let $T = \dfrac{X}{2} + \dfrac{Y}{2}$.

(c) (i) Show that T is an unbiased estimator for λ.

(ii) Show that T is a more efficient unbiased estimator of λ than S. [3]

(d) Could either S or T model a Poisson distribution? Justify your answer. [1]

5. [Maximum mark: 10]

Two species of plant, A and B, are identical in appearance though it is known that the mean length of leaves from a plant of species A is $5.2\,\text{cm}$, whereas the mean length of leaves from a plant of species B is $4.6\,\text{cm}$. Both lengths can be modelled by normal distributions with standard deviation $1.2\,\text{cm}$.

In order to test whether a particular plant is from species A or species B, 16 leaves are collected at random from the plant. The length, x, of each leaf is measured and the mean length evaluated. A one-tailed test of the sample mean, $\bar{X}$, is then performed at the 5% level, with the hypotheses: $H_0 : \mu = 5.2$ and $H_1 : \mu < 5.2$.

(a) Find the critical region for this test. [3]

(b) Find the probability of a Type II error if the leaves are in fact from a plant of species B. [2]

It is now known that in the area in which the plant was found 90% of all the plants are of species A and 10% are of species B.

(c) Find the probability that $\bar{X}$ will fall within the critical region of the test. [2]

(d) If, having done the test, the sample mean is found to lie within the critical region, find the probability that the leaves came from a plant of species A. [3]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

Specimen (adapted from November 2014)

Statistics and probability

Higher level

Paper 3

10 pages

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

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A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

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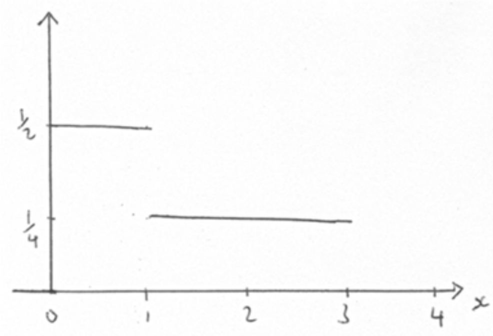
1. (a)



A1

Note: Ignore open / closed endpoints and vertical lines.

Note: Award ***A1*** for a correct graph with scales on both axes and a clear indication of the relevant values.

[1 mark]

(b)
$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{2} & 0 \le x < 1 \\ \dfrac{x}{4} + \dfrac{1}{4} & 1 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

considering the areas in their sketch or using integration ***(M1)***

$F(x) = 0, \quad x < 0$, $F(x) = 1, \quad x \ge 3$ ***A1***

$F(x) = \dfrac{x}{2}, \quad 0 \le x < 1$ ***A1***

$F(x) = \dfrac{x}{4} + \dfrac{1}{4}, \quad 1 \le x < 3$ ***A1A1***

Note: Accept $<$ for $\le$ in all places and also $>$ for $\ge$ first ***A1***.

[5 marks]

(c) $Q_3 = 2$ ***A1***

[1 mark]

Total [7 marks]

2. (a) **METHOD 1**

let X be the number of throws until Eric hits the target three times

$X \sim \mathrm{NB}(3, 0.2)$ ***(M1)***

$\mathrm{P}(X = 6) = \binom{5}{2} 0.8^3 \times 0.2^3$ ***(A1)***

$= 0.04096 \left(= \frac{128}{3125} \right)$ (exact) ***A1***

METHOD 2

let X be the number of hits in five throws

X is $\mathrm{B}(5, 0.2)$ ***(M1)***

$\mathrm{P}(X = 2) = \binom{5}{2} 0.2^3 \times 0.8^3 \quad (0.2048)$ ***(A1)***

P(3rd hit on 6th throw) $= \binom{5}{2} 0.2^2 \times 0.8^3 \times 0.2 = 0.04096 \left(= \frac{128}{3125} \right)$ (exact) ***A1***

[3 marks]

(b) (i) expected number of throws $= \frac{3}{0.2} = 15$ ***(M1)A1***

(ii) profit $= (10 - 15) = -\$5$ or loss $= \$5$ ***A1***

[3 marks]

(c) **METHOD 1**

let Y be the number of times the target is hit in 8 throws

$Y \sim \mathrm{B}(8, 0.2)$ ***(M1)***

$\mathrm{P}(Y \le 2)$ ***(M1)***

$= 0.797$ ***A1***

METHOD 2

let the 3rd hit occur on the Yth throw

Y is $\mathrm{NB}(3, 0.2)$ ***(M1)***

$\mathrm{P}(Y > 8) = 1 - \mathrm{P}(Y \le 8)$ ***(M1)***

$= 0.797$ ***A1***

[3 marks]

Total [9 marks]

3. (a) **METHOD 1**

$\mathrm{Cov}(X, Y) = \mathrm{E}\left((X-\mu_X)(Y-\mu_Y)\right)$

$= \mathrm{E}\left(XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y\right)$ ***(M1)***

$= \mathrm{E}(XY) - \mu_Y \mathrm{E}(X) - \mu_X \mathrm{E}(Y) + \mu_X\mu_Y$

$= \mathrm{E}(XY) - \mu_X\mu_Y$ ***A1***

as X and Y are independent $\mathrm{E}(XY) = \mu_X\mu_Y$ ***R1***

$\mathrm{Cov}(X, Y) = 0$ ***AG***

METHOD 2

$\mathrm{Cov}(X, Y) = \mathrm{E}\left((X-\mu_X)(Y-\mu_Y)\right)$

$= \mathrm{E}\left(X-\mu_x\right)\mathrm{E}\left(Y-\mu_y\right)$ ***(M1)***

since X, Y are independent ***R1***

$= (\mu_x - \mu_x)(\mu_y - \mu_y)$ ***A1***

$= 0$ ***AG***

[3 marks]

(b) $H_0 : \rho = 0 \quad H_1 : \rho < 0$ ***A1***

Note: The hypotheses must be expressed in terms of ρ.

test statistic $t_{test} = -0.35\sqrt{\dfrac{20-2}{1-(-0.35)^2}}$ ***(M1)(A1)***

$= -1.585\ldots$ ***(A1)***

degrees of freedom $= 18$ ***(A1)***

EITHER

p-value $= 0.0652$ ***A1***

this is greater than 0.05 ***M1***

OR

$t_{5\,\%}(18) = -1.73$ ***A1***

this is less than -1.59 ***M1***

THEN

hence accept H_0 or reject H_1 or equivalent or contextual equivalent ***R1***

Note: Allow follow through for the final ***R1*** mark.

[8 marks]

Total [11 marks]

4. (a) (i) $G'(t) = \lambda e^{\lambda(t-1)}$ ***A1***

$\mathrm{E}(X) = G'(1)$ ***M1***

$= \lambda$ ***AG***

(ii) $G''(t) = \lambda^2 e^{\lambda(t-1)}$ ***M1***

$\Rightarrow G''(1) = \lambda^2$ ***(A1)***

$\mathrm{Var}(X) = G''(1) + G'(1) - \left(G'(1)\right)^2$ ***(M1)***

$= \lambda^2 + \lambda - \lambda^2$ ***A1***

$= \lambda$ ***AG***

[6 marks]

(b) (i) $\mathrm{E}(S) = 2\lambda - \lambda = \lambda$ ***A1***

(ii) $\mathrm{Var}(S) = 4\lambda + \lambda = 5\lambda$ ***(A1)A1***

Note: First ***A1*** can be awarded for either 4λ or $+\lambda$.

[3 marks]

(c) (i) $\mathrm{E}(T) = \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda$ (so T is an unbiased estimator) ***A1***

(ii) $\mathrm{Var}(T) = \frac{1}{4}\lambda + \frac{1}{4}\lambda = \frac{1}{2}\lambda$ ***A1***

this is less than $\mathrm{Var}(S)$, therefore T is the more efficient estimator ***R1AG***

Note: Follow through their variances from (b)(ii) and(c)(ii).

[3 marks]

(d) no, mean does not equal the variance ***R1***

[1 mark]

Total [13 marks]

5. (a) $\bar{X} \sim N\left(5.2, \frac{1.2^2}{16}\right)$ ***(M1)***

critical value is $5.2 - 1.64485\ldots \times \frac{1.2}{4} = 4.70654\ldots$ ***(A1)***

critical region is $\left]-\infty, 4.71\right]$ ***A1***

Note: Allow follow through for the final ***A1*** from their critical value.

[3 marks]

Note: Follow through previous values in (b), (c) and (d).

(b) type II error probability $= \mathrm{P}\left(\bar{X} > 4.70654\ldots \middle| \bar{X} \text{ is } N\left(4.6, \frac{1{,}2^2}{16}\right)\right)$ ***(M1)***

$= 0.361$ ***A1***

[2 marks]

(c) $0.9 \times 0.05 + 0.1 \times (1 - 0.361\ldots) = 0.108875997\ldots = 0.109$ ***M1A1***

Note: Award ***M1*** for a weighted average of probabilities with weights $0.1, 0.9$.

[2 marks]

(d) attempt to use conditional probability formula ***M1***

$\frac{0.9 \times 0.05}{0.108875997\ldots}$ ***(A1)***

$= 0.41334\ldots = 0.413$ ***A1***

[3 marks]

Total [10 marks]