International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

Further mathematics
Higher level

Specimen papers 1 and 2

For first examinations in 2014

CONTENTS

International Baccalaureate®
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FURTHER MATHEMATICS
HIGHER LEVEL
PAPER 1

SPECIMEN

2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[150 marks]*.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 6]*

Using l'Hôpital's Rule, determine the value of

$$\lim_{x\to 0}\frac{\tan x - x}{1-\cos x}.$$

2. *[Maximum mark: 8]*

(a) Show that the following vectors form a basis for the vector space $\mathbb{R}^3$.

$$\begin{pmatrix}1\\2\\3\end{pmatrix};\begin{pmatrix}2\\3\\1\end{pmatrix};\begin{pmatrix}5\\2\\5\end{pmatrix}$$ *[3 marks]*

(b) Express the following vector as a linear combination of the above vectors.

$$\begin{pmatrix}12\\14\\16\end{pmatrix}$$ *[5 marks]*

3. *[Maximum mark: 10]*

The positive integer N is represented by 4064 in base b and 2612 in base $b+1$.

(a) Determine the value of b. *[4 marks]*

(b) Find the representation of N

(i) in base 10;

(ii) in base 12. *[6 marks]*

4. *[Maximum mark: 11]*

The weights of potatoes in a shop are normally distributed with mean 98 grams and standard deviation 16 grams.

(a) The shopkeeper places 100 randomly chosen potatoes on a weighing machine. Find the probability that their total weight exceeds 10 kilograms. *[3 marks]*

(b) Find the minimum number of randomly selected potatoes which are needed to ensure that their total weight exceeds 10 kilograms with probability greater than 0.95. *[8 marks]*

5. *[Maximum mark: 11]*

(a) The point $\mathrm{T}(at^2, 2at)$ lies on the parabola $y^2 = 4ax$. Show that the tangent to the parabola at T has equation $y = \frac{x}{t} + at$. *[3 marks]*

(b) The distinct points $\mathrm{P}(ap^2, 2ap)$ and $\mathrm{Q}(aq^2, 2aq)$, where $p, q \neq 0$, also lie on the parabola $y^2 = 4ax$. Given that the line (PQ) passes through the focus, show that

(i) $pq = -1$;

(ii) the tangents to the parabola at P and Q, intersect on the directrix. *[8 marks]*

6. *[Maximum mark: 7]*

The group $\{G, +\}$ is defined by the operation of addition on the set $G = \{2n \mid n \in \mathbb{Z}\}$.
The group $\{H, +\}$ is defined by the operation of addition on the set $H = \{4n \mid n \in \mathbb{Z}\}$.
Prove that $\{G, +\}$ and $\{H, +\}$ are isomorphic.

7. *[Maximum mark: 9]*

(a) Given that $a \equiv b \pmod p$, show that $a^n \equiv b^n \pmod p$ for all $n \in \mathbb{Z}^+$. *[4 marks]*

(b) Show that $29^{13} + 13^{29}$ is exactly divisible by 7. *[5 marks]*

8. *[Maximum mark: 8]*

Consider the infinite series $S = \sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(\frac{1}{n}\right)$.

(a) Show that the series is conditionally convergent but not absolutely convergent. *[6 marks]*

(b) Show that $S > 0.4$. *[2 marks]*

9. *[Maximum mark: 8]*

Consider the system of equations

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 2 & -1 \\ 3 & 5 & -4 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \\ k \end{pmatrix}.$$

(a) By reducing the augmented matrix to row echelon form,

(i) find the rank of the coefficient matrix;

(ii) find the value of k for which the system has a solution. *[5 marks]*

(b) For this value of k, determine the solution. *[3 marks]*

10. *[Maximum mark: 11]*

Bill is investigating whether or not there is a positive association between the heights and weights of boys of a certain age. He defines the hypotheses

$$H_0 : \rho = 0;\ H_1 : \rho > 0,$$

where ρ denotes the population correlation coefficient between heights and weights of boys of this age. He measures the height, h cm, and weight, w kg, of each of a random sample of 20 boys of this age and he calculates the following statistics.

$$\sum w = 340, \sum h = 2002, \sum w^2 = 5830, \sum h^2 = 201124, \sum hw = 34150$$

(This question continues on the following page)

(Question 10 continued)

(a) (i) Calculate the correlation coefficient for this sample.

(ii) Calculate the p-value of your result and interpret it at the 1 % level of significance. *[8 marks]*

(b) (i) Calculate the equation of the least squares regression line of w on h.

(ii) The height of a randomly selected boy of this age of 90 cm. Estimate his weight. *[3 marks]*

11. *[Maximum mark: 11]*

The function f is defined by $f(x) = \mathrm{e}^x \cos x$.

(a) Show that $f''(x) = -2\mathrm{e}^x \sin x$. *[2 marks]*

(b) Determine the Maclaurin series for $f(x)$ up to and including the term in x^4. *[5 marks]*

(c) By differentiating your series, determine the Maclaurin series for $\mathrm{e}^x \sin x$ up to the term in x^3. *[4 marks]*

12. *[Maximum mark: 14]*

The matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 4 & 1 \\ -4 & -11 & -2 \end{pmatrix}.$$

(a) (i) Find the matrices $\boldsymbol{A}^2$ and $\boldsymbol{A}^3$, and verify that $\boldsymbol{A}^3 = 2\boldsymbol{A}^2 - \boldsymbol{A}$.

(ii) Deduce that $\boldsymbol{A}^4 = 3\boldsymbol{A}^2 - 2\boldsymbol{A}$. *[6 marks]*

(b) (i) Suggest a similar expression for $\boldsymbol{A}^n$ in terms of $\boldsymbol{A}$ and $\boldsymbol{A}^2$, valid for $n \geq 3$.

(ii) Use mathematical induction to prove the validity of your suggestion. *[8 marks]*

13. *[Maximum mark: 9]*

A sequence $\{u_n\}$ satisfies the recurrence relation $u_{n+2} = 2u_{n+1} - 5u_n$, $n \in \mathbb{N}$. Obtain an expression for u_n in terms of n given that $u_0 = 0$ and $u_1 = 1$.

14. *[Maximum mark: 13]*

The set S contains the eight matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

where a, b, c can each take one of the values $+1$ or -1.

(a) Show that any matrix of this form is its own inverse. *[3 marks]*

(b) Show that S forms an Abelian group under matrix multiplication. *[9 marks]*

(c) Giving a reason, state whether or not this group is cyclic. *[1 mark]*

15. *[Maximum mark: 14]*

(a) Prove the internal angle bisector theorem, namely that the internal bisector of an angle of a triangle divides the side opposite the angle into segments proportional to the sides adjacent to the angle. *[6 marks]*

(b) The bisector of the exterior angle $\hat{A}$ of the triangle ABC meets (BC) at P. The bisector of the interior angle $\hat{B}$ meets [AC] at Q. Given that (PQ) meets [AB] at R, use Menelaus' theorem to prove that (CR) bisects the angle $\mathrm{A\hat{C}B}$. *[8 marks]*

SPEC/5/FURMA/HP1/ENG/TZX/XX/M

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MARKSCHEME

SPECIMEN

FURTHER MATHEMATICS

Higher Level

Paper 1

16 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) are often dependent on the preceding ***M*** mark.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.* do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). Apply a **MR** penalty of 1 mark to that question. Award the marks as usual and then write –1(**MR**) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled **(d)** and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates may not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

The method of dealing with accuracy errors on a whole paper basis by means of the Accuracy Penalty ***(AP)*** *no longer applies.*

Instructions to examiners about such numerical issues will be provided on a question by question basis within the framework of mathematical correctness, numerical understanding and contextual appropriateness.

The rubric on the front page of each question paper is given for the guidance of candidates. The markscheme (MS) may contain instructions to examiners in the form of "Accept answers which round to n significant figures (sf)". Where candidates state answers, required by the question, to fewer than n sf, award ***A0****. Some intermediate numerical answers may be required by the MS but not by the question. In these cases only award the mark(s) if the candidate states the answer exactly or to at least 2sf.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (e.g. TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. $\lim\limits_{x \to 0} \dfrac{\tan x - x}{1 - \cos x} = \lim\limits_{x \to 0} \dfrac{\sec^2 x - 1}{\sin x}$ ***M1A1A1***

this still gives $\dfrac{0}{0}$

EITHER

repeat the process ***M1***

$= \lim\limits_{x \to 0} \dfrac{2\sec^2 x \tan x}{\cos x}$ ***A1***

$= 0$ ***A1***

OR

$= \lim\limits_{x \to 0} \dfrac{\tan^2 x}{\sin x}$ ***M1***

$= \lim\limits_{x \to 0} \dfrac{\sin x}{\cos^2 x}$ ***A1***

$= 0$ ***A1***

[6 marks]

2. (a) let $\boldsymbol{A} = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 2 \\ 3 & 1 & 5 \end{vmatrix}$ and consider $\det(\boldsymbol{A}) = -30$ ***(M1)A1***

the vectors form a basis because the determinant is non-zero (or because the matrix is non-singular) ***R1***

[3 marks]

(b) let $\begin{pmatrix} 12 \\ 14 \\ 16 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \nu \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$ ***M1A1***

so that

EITHER

$\lambda + 2\mu + 5\nu = 12$

$2\lambda + 3\mu + 2\nu = 14$ ***M1***

$3\lambda + \mu + 5\nu = 16$

OR

$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 2 \\ 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} = \begin{pmatrix} 12 \\ 14 \\ 16 \end{pmatrix}$ ***M1***

THEN

giving $\lambda = 3,\ \mu = 2,\ \nu = 1$ ***(A1)***

hence $\begin{pmatrix} 12 \\ 14 \\ 16 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$ ***A1***

[5 marks]

Total [8 marks]

3. (a) the equation satisfied by b is

$4b^3+6b+4=2(b+1)^3+6(b+1)^2+(b+1)+2$ ***M1A1***

$2b^3-12b^2-13b-7=0$ ***(A1)***

$b=7$ ***A1***

[4 marks]

(b) (i) $N=4\times 7^3+6\times 7+4=1418$

or $N=2\times 8^3+6\times 8^2+1\times 8+2=1418$ ***(M1)A1***

(ii) $12\underline{\rvert 1418}$ ***(M1)***

$12\underline{\rvert 118}$ remainder 2 ***(A1)***

$12\underline{\rvert 9}$ remainder A (where $\text{A}=10$) ***(A1)***

$1418=(9\text{A}2)_{12}$ ***A1***

Note: Accept alternative correct methods.

[6 marks]

Total [10 marks]

4. (a) let T denote the total weight, then

$T\sim \text{N}(9800,\ 25600)$ ***(M1)(A1)***

$\text{P}(T>10000)=0.106$ ***A1***

[3 marks]

(b) let there be n potatoes, in this case,

$T\sim \text{N}(98n,\ 256n)$ ***A1***

we require

$\text{P}(T>10000)>0.95$ ***(M1)***

or equivalently

$\text{P}(T\le 10000)<0.05$ ***A1***

standardizing,

$\text{P}\left(Z\le \dfrac{10000-98n}{16\sqrt{n}}\right)<0.05$ ***A1***

$\dfrac{10000-98n}{16\sqrt{n}}<-1.6449\ldots$ ***(A1)***

$98n-26.32\sqrt{n}-10000>0$ ***A1***

solving the corresponding equation, $n=104.7\ldots$ ***(A1)***

the required minimum value is 105 ***A1***

Note: Part (b) could also be solved using SOLVER and normalcdf, or by trial and improvement.

Note: Allow the use of $=$ instead of $<$ and $>$ throughout.

[8 marks]

Total [11 marks]

5. (a) $2y\frac{dy}{dx} = 4a$ ***M1***

$\frac{dy}{dx} = \frac{2a}{y} = \frac{1}{t}$ ***A1***

Note: Accept parametric differentiation.

the equation of the tangent is

$y - 2at = \frac{1}{t}(x - at^2)$ ***A1***

$y = \frac{x}{t} + at$ ***AG***

Note: Accept equivalent based on $y = mx + c$.

[3 marks]

(b) (i) the focus F is $(a, 0)$ ***A1***

EITHER

the gradient of (PQ) is $\frac{2a(p-q)}{a(p^2-q^2)} = \frac{2}{p+q}$ ***M1A1***

the equation of (PQ) is $y = \frac{2x}{p+q} + \frac{2apq}{p+q}$ ***A1***

substitute $x = a$, $y = 0$ ***M1***

$pq = -1$ ***AG***

OR

the condition for PFQ to be collinear is

$\frac{2a(p-q)}{a(p^2-q^2)} = \frac{2ap}{ap^2-a}$ ***M1A1***

$\frac{2}{p+q} = \frac{2p}{p^2-1}$ ***A1***

$p^2 - 1 = p^2 + pq$ ***A1***

$pq = -1$ ***AG***

Note: There are alternative approaches.

(ii) the equations of the tangents at P and Q are

$y = \frac{x}{p} + ap$ and $y = \frac{x}{q} + aq$

the tangents meet where

$\frac{x}{p} + ap = \frac{x}{q} + aq$ ***M1***

$x = apq = -a$ ***A1***

the equation of the directrix is $x = -a$ ***R1***

so that the tangents meet on the directrix ***AG***

[8 marks]

Total [11 marks]

6. consider the function $f: G \to H$ defined by $f(g) = 2g$ where $g \in G$ ***A1***

given $g_1, g_2 \in G$, $f(g_1) = f(g_2) \Rightarrow 2g_1 = 2g_2 \Rightarrow g_1 = g_2$ (injective) ***M1***

given $h \in H$ then $h = 4n$, so $f(2n) = h$ and $2n \in G$ (surjective) ***M1***

hence f is a bijection ***A1***

then, for $g_1, g_2 \in G$

$f(g_1 + g_2) = 2(g_1 + g_2)$ ***A1***

$f(g_1) + f(g_2) = 2g_1 + 2g_2$ ***A1***

it follows that $f(g_1 + g_2) = f(g_1) + f(g_2)$ ***R1***

which completes the proof that $\{G, +\}$ and $\{H, +\}$ are isomorphic ***AG***

[7 marks]

7. (a) $a \equiv b(\text{mod } p) \Rightarrow a = b + pN$, $N \in \mathbb{Z}$ ***M1***

$a^n = (b + pN)^n = b^n + nb^{n-1}pN \ldots$ ***M1A1***

$= b^n + pM$ where $M \in \mathbb{Z}$ ***A1***

this shows that $a^n \equiv b^n(\text{mod } p)$ ***AG***

[4 marks]

(b) $29 \equiv 1(\text{mod } 7) \Rightarrow 29^{13} \equiv 1^{13} \equiv 1(\text{mod } 7)$ ***M1A1***

$13 \equiv -1(\text{mod } 7) \Rightarrow 13^{29} \equiv (-1)^{29} \equiv -1(\text{mod } 7)$ ***A1***

therefore $29^{13} + 13^{29} \equiv 1 + (-1) \equiv 0(\text{mod } 7)$ ***M1A1***

this shows that $29^{13} + 13^{29}$ is exactly divisible by 7 ***AG***

[5 marks]

Total [9 marks]

8. (a) $\sin\left(\frac{1}{n}\right)$ decreases as n increases ***A1***

$\sin\left(\frac{1}{n}\right) \to 0$ as $n \to \infty$ ***A1***

so using the alternating series test, the series is conditionally convergent ***R1***

comparing (the absolute series) with the harmonic series:

$\sum_{n=1}^{\infty} \frac{1}{n}$ ***M1***

$\lim_{n \to \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$ ***A1***

since the harmonic series is divergent, it follows by the limit comparison theorem that the given series is not absolutely convergent ***R1***

hence the series is conditionally convergent ***AG***

[6 marks]

continued …

Question 8 continued

(b) successive partial sums are
0.841…
0.362…
0.689…
0.441… ***A1***
since S lies between any pair of successive partial sums, it follows that S lies between 0.441… and 0.689… ***R1***
and is therefore greater than 0.4 ***AG***

Note: Use of the facts that the error is always less than the modulus of the next term, or the sequence of even partial sums gives lower bounds are equally acceptable.

[2 marks]

Total [8 marks]

9. (a) reducing to row echelon form

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 4 & -5 & -7 \\ 0 & 8 & -10 & -14 \\ 0 & 4 & -5 & k-15 \end{array}\right)$$ ***(M1)(A1)***

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 4 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k-8 \end{array}\right)$$ ***A1***

(i) this shows that the rank of the matrix is 2 ***A1***

(ii) the equations can be solved if $k = 8$ ***A1***

[5 marks]

(b) let $z = \lambda$ ***A1***

then $y = \dfrac{5\lambda - 7}{4}$ ***A1***

and $x = \left(5 - 2\lambda + \dfrac{5\lambda - 7}{4} = \right) \dfrac{13 - 3\lambda}{4}$ ***A1***

Note: Accept equivalent expressions.

[3 marks]

Total [8 marks]

10. (a) (i) $r = \dfrac{34150 - 340 \times \dfrac{2002}{20}}{\sqrt{\left(5830 - \dfrac{340^2}{20}\right)\left(201124 - \dfrac{2002^2}{20}\right)}}$ ***(M1)(A1)***

Note: Accept equivalent formula.

$= 0.610$ ***A1***

(ii) $\left(T = R \times \sqrt{\dfrac{n-2}{1-R^2}} \text{ has the } t\text{-distribution with } n-2 \text{ degrees of freedom}\right)$

$t = 0.6097666\ldots\sqrt{\dfrac{18}{1-0.6097666\ldots^2}}$ ***M1***

$= 3.2640\ldots$ ***A1***

$\text{DF} = 18$ ***A1***

p-value $= 0.00215\ldots$ ***A1***

this is less than 0.01, so we conclude that there is a positive association between heights and weights of boys of this age ***R1***

[8 marks]

(b) (i) the equation of the regression line of w on h is

$w - \dfrac{340}{20} = \left(\dfrac{20 \times 34150 - 340 \times 2002}{20 \times 201124 - 2002^2}\right)\left(h - \dfrac{2002}{20}\right)$ ***M1***

$w = 0.160h + 0.957$ ***A1***

(ii) putting $h = 90$, $w = 15.4$ (kg) ***A1***

Note: Award ***M0A0A0*** for calculation of h on w.

[3 marks]

Total [11 marks]

11. (a) $f'(x) = e^x \cos x - e^x \sin x$ ***A1***

$f''(x) = e^x \cos x - e^x \sin x - e^x \sin x - e^x \cos x$ ***A1***

$= -2e^x \sin x$ ***AG***

[2 marks]

(b) $f'''(x) = -2e^x \sin x - 2e^x \cos x$ ***A1***

$f^{IV}(x) = -4e^x \cos x$ ***A1***

$f(0) = 1,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -2,\ f^{IV}(0) = -4$ ***(A1)***

the Maclaurin series is

$e^x \cos x = 1 + x - \dfrac{x^3}{3} - \dfrac{x^4}{6} + \ldots$ ***M1A1***

Note: Accept multiplication of series method.

[5 marks]

continued …

Question 11 continued

(c) differentiating,

$$e^x \cos x - e^x \sin x = 1 - x^2 - \frac{2x^3}{3} + \ldots$$ ***M1A1***

$$e^x \sin x = 1 + x - \frac{x^3}{3} \ldots - 1 + x^2 + \frac{2x^3}{3} + \ldots$$ ***M1***

$$= x + x^2 + \frac{x^3}{3} + \ldots$$ ***A1***

[4 marks]

Total [11 marks]

12. (a) (i) $$\boldsymbol{A}^2 = \begin{pmatrix} 2 & 4 & 1 \\ 4 & 7 & 2 \\ -14 & -26 & -7 \end{pmatrix}$$ ***A1***

$$\boldsymbol{A}^3 = \begin{pmatrix} 4 & 7 & 2 \\ 6 & 10 & 3 \\ -24 & -41 & -12 \end{pmatrix}$$ ***A1***

$$2\boldsymbol{A}^2 - \boldsymbol{A} = 2\begin{pmatrix} 2 & 4 & 1 \\ 4 & 7 & 2 \\ -14 & -26 & -7 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 2 & 4 & 1 \\ -4 & -11 & -2 \end{pmatrix}$$ ***M1***

$$= \begin{pmatrix} 4 & 7 & 2 \\ 6 & 10 & 3 \\ -24 & -41 & -12 \end{pmatrix} = \boldsymbol{A}^3$$ ***AG***

(ii) $\boldsymbol{A}^4 = \boldsymbol{A}\boldsymbol{A}^3$ ***M1***

$= \boldsymbol{A}(2\boldsymbol{A}^2 - \boldsymbol{A})$ ***A1***

$= 2\boldsymbol{A}^3 - \boldsymbol{A}^2$

$= 2(2\boldsymbol{A}^2 - \boldsymbol{A}) - \boldsymbol{A}^2$ ***A1***

$= 3\boldsymbol{A}^2 - 2\boldsymbol{A}$ ***AG***

Note:	Accept alternative solutions that include correct calculation of both sides of the expression.

[6 marks]

continued …

Question 12 continued

(b) (i) conjecture: $\boldsymbol{A}^n = (n-1)\boldsymbol{A}^2 - (n-2)\boldsymbol{A}$ ***A1***

(ii) first check that the result is true for $n = 3$
the formula gives $\boldsymbol{A}^3 = 2\boldsymbol{A}^2 - \boldsymbol{A}$ which is correct ***A1***
assume the result for $n = k$, *i.e.* ***M1***
$\boldsymbol{A}^k = (k-1)\boldsymbol{A}^2 - (k-2)\boldsymbol{A}$
so
$\boldsymbol{A}^{k+1} = \boldsymbol{A}[(k-1)\boldsymbol{A}^2 - (k-2)\boldsymbol{A}]$ ***M1***
$= (k-1)\boldsymbol{A}^3 - (k-2)\boldsymbol{A}^2$ ***A1***
$= (k-1)(2\boldsymbol{A}^2 - \boldsymbol{A}) - (k-2)\boldsymbol{A}^2$ ***M1***
$= k\boldsymbol{A}^2 - (k-1)\boldsymbol{A}$ ***A1***
so true for $n = k \Rightarrow$ true for $n = k+1$ and since true for $n = 3$,
the result is proved by induction ***R1***

Note:	Only award the ***R1*** mark if a reasonable attempt at a proof by induction has been made.

[8 marks]

Total [14 marks]

13. the auxiliary equation is $m^2 - 2m + 5 = 0$ ***M1***
the roots are $1 \pm 2\mathrm{i}$ ***A1***
the general solution is $u_n = A(1+2\mathrm{i})^n + B(1-2\mathrm{i})^n$ ***A1***
substituting $u_0 = 0$ and $u_1 = 1$ ***M1***
$A + B = 0$
$A(1+2\mathrm{i}) + B(1-2\mathrm{i}) = 1$ ***A1***

Note:	Only award above ***A1*** if both equations are correct.

solving
$A(1+2\mathrm{i} - 1 + 2\mathrm{i}) = 1$ ***(M1)***
$A = \frac{1}{4\mathrm{i}}$ or $-\frac{\mathrm{i}}{4}$ ***A1***
$B = -\frac{1}{4\mathrm{i}}$ or $\frac{\mathrm{i}}{4}$ ***A1***
therefore
$u_n = \frac{1}{4\mathrm{i}}(1+2\mathrm{i})^n - \frac{1}{4\mathrm{i}}(1-2\mathrm{i})^n$ or $\frac{\mathrm{i}}{4}(1-2\mathrm{i})^n - \frac{\mathrm{i}}{4}(1+2\mathrm{i})^n$ ***A1***

[9 marks]

14. (a) $$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$$ ***M1A1***

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ ***A1***

this shows that each matrix is self-inverse

[3 marks]

(b) closure:

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{pmatrix}\begin{pmatrix} a_2 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & 0 & 0 \\ 0 & b_1b_2 & 0 \\ 0 & 0 & c_1c_2 \end{pmatrix}$$ ***M1A1***

$$= \begin{pmatrix} a_3 & 0 & 0 \\ 0 & b_3 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$$

where each of a_3, b_3, c_3 can only be ± 1 ***A1***

this proves closure

identity: the identity matrix is the group identity ***A1***

inverse: as shown above, every element is self-inverse ***A1***

associativity: this follows because matrix multiplication is associative ***A1***

S is therefore a group ***AG***

Abelian: $$\begin{pmatrix} a_2 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_2 \end{pmatrix}\begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{pmatrix} = \begin{pmatrix} a_2a_1 & 0 & 0 \\ 0 & b_2b_1 & 0 \\ 0 & 0 & c_2c_1 \end{pmatrix}$$ ***A1***

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{pmatrix}\begin{pmatrix} a_2 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & 0 & 0 \\ 0 & b_1b_2 & 0 \\ 0 & 0 & c_1c_2 \end{pmatrix}$$ ***A1***

Note: Second line may have been shown whilst proving closure, however a reference to it must be made here.

we see that the same result is obtained either way which proves commutativity so that the group is Abelian ***R1***

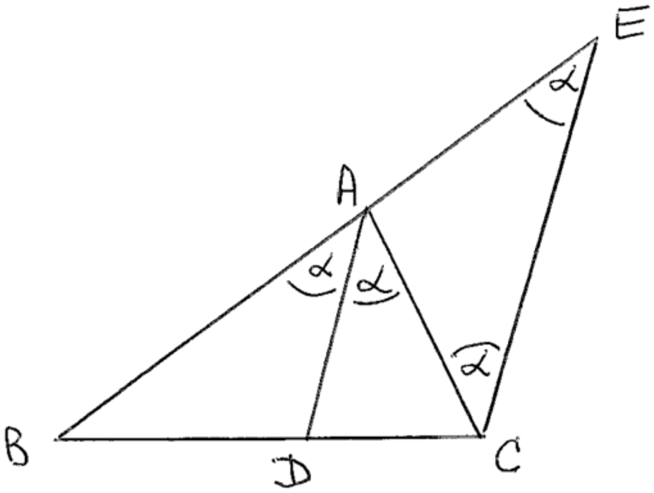
[9 marks]

(c) since all elements (except the identity) are of order 2, the group is not cyclic (since *S* contains 8 elements) ***R1***

[1 mark]

Total [13 marks]

15. (a) **EITHER**



let [AD] bisect A, draw a line through C parallel to (AD) meeting (AB) at E ***M1***

then $\text{B}\hat{\text{A}}\text{D} = \text{A}\hat{\text{E}}\text{C}$ and $\text{D}\hat{\text{A}}\text{C} = \text{A}\hat{\text{C}}\text{E}$ ***A1***

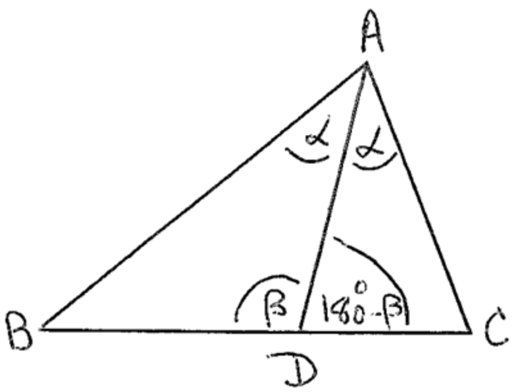
since $\text{B}\hat{\text{A}}\text{D} = \text{D}\hat{\text{A}}\text{C}$ it follows that $\text{A}\hat{\text{E}}\text{C} = \text{A}\hat{\text{C}}\text{E}$ ***A1***

triangle AEC is therefore isosceles and $\text{AE} = \text{AC}$ ***A1***

since triangles BAD and BEC are similar

$\frac{\text{BD}}{\text{DC}} = \frac{\text{AB}}{\text{AE}} = \frac{\text{AB}}{\text{AC}}$ ***M1A1***

OR



$\frac{\text{AB}}{\sin\beta} = \frac{\text{BD}}{\sin\alpha}$ ***M1A1***

$\frac{\text{AC}}{\sin(180-\beta)} = \frac{\text{DC}}{\sin\alpha}$ ***M1A1***

$\sin\beta = \sin(180-\beta)$ ***R1***

$\Rightarrow \frac{\text{AB}}{\text{BD}} = \frac{\text{AC}}{\text{DC}}$

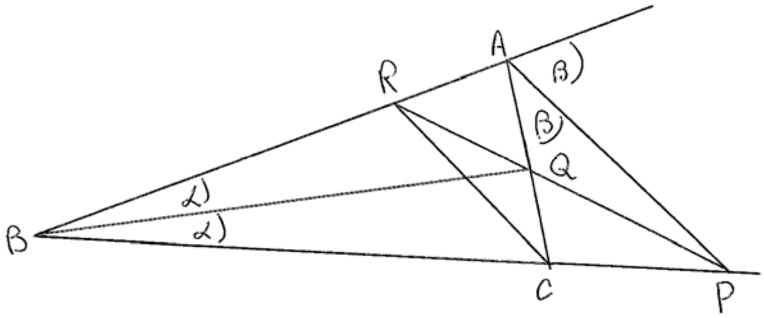
$\Rightarrow \frac{\text{BD}}{\text{DC}} = \frac{\text{AB}}{\text{AC}}$ ***A1***

[6 marks]

continued …

Question 15 continued

(b)



using the angle bisector theorem, ***M1***

$\dfrac{AQ}{QC} = \dfrac{AB}{BC}$ and $\dfrac{BP}{PC} = \dfrac{AB}{AC}$ ***A1***

using Menelaus' theorem with (PR) as transversal to triangle ABC ***M1***

$\dfrac{BR}{AR} \times \dfrac{AQ}{QC} \times \dfrac{PC}{BP} = (-)1$ ***A1***

substituting the above results, ***M1***

$\dfrac{BR}{AR} \times \dfrac{AB}{BC} \times \dfrac{AC}{AB} = (-)1$ ***A1***

giving

$\dfrac{BR}{AR} = \dfrac{BC}{AC}$ ***A1***

[CR] therefore bisects angle C by (the converse to) the angle bisector theorem ***R1AG***

[8 marks]

Total [14 marks]

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FURTHER MATHEMATICS
HIGHER LEVEL
PAPER 2

SPECIMEN

2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[150 marks]*.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 9]*

The relation R is defined on $\mathbb{R}^+ \times \mathbb{R}^+$ such that $(x_1, y_1) R (x_2, y_2)$ if and only if $\frac{x_1}{x_2} = \frac{y_2}{y_1}$.

(a) Show that R is an equivalence relation. *[6 marks]*

(b) Determine the equivalence class containing (x_1, y_1) and interpret it geometrically. *[3 marks]*

2. *[Maximum mark: 10]*

The weights of apples, in grams, produced on a farm may be assumed to be normally distributed with mean μ and variance σ^2.

(a) The farm manager selects a random sample of 10 apples and weighs them with the following results, given in grams.

82, 98, 102, 96, 111, 95, 90, 89, 99, 101

(i) Determine unbiased estimates for μ and σ^2.

(ii) Determine a 95 % confidence interval for μ. *[5 marks]*

(b) The farm manager claims that the mean weight of apples is 100 grams but the buyer from the local supermarket claims that the mean is less than this. To test these claims, they select a random sample of 100 apples and weigh them. Their results are summarized as follows, where x is the weight of an apple in grams.

$$\sum x = 9831; \sum x^2 = 972578$$

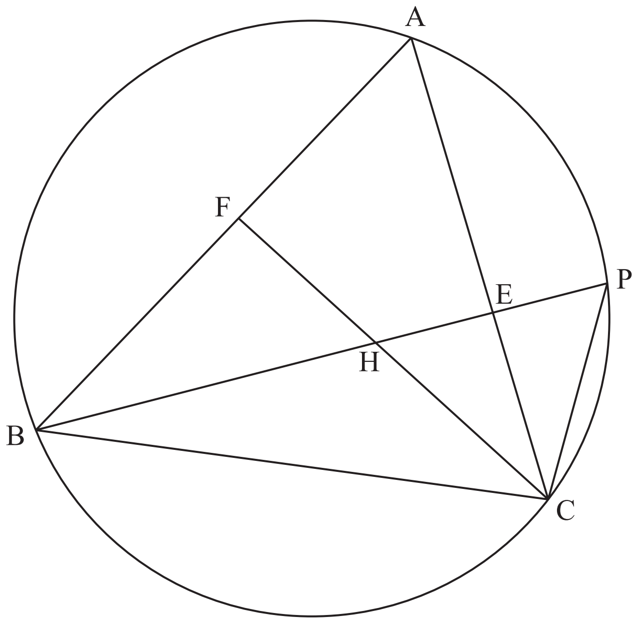
(i) State suitable hypotheses for testing these claims.

(ii) Determine the p-value for this test.

(iii) At the 1 % significance level, state which claim you accept and justify your answer. *[5 marks]*

3. *[Maximum mark: 15]*

In the acute angled triangle ABC, the points E, F lie on [AC], [AB] respectively such that [BE] is perpendicular to [AC] and [CF] is perpendicular to [AB]. The lines (BE) and (CF) meet at H. The line (BE) meets the circumcircle of the triangle ABC at P. This is shown in the following diagram.



(a) (i) Show that CEFB is a cyclic quadrilateral.

(ii) Show that $\text{HE} = \text{EP}$. *[7 marks]*

The line (AH) meets [BC] at D.

(b) (i) By considering cyclic quadrilaterals show that $\text{C}\hat{\text{A}}\text{D} = \text{E}\hat{\text{F}}\text{H} = \text{E}\hat{\text{B}}\text{C}$.

(ii) Hence show that [AD] is perpendicular to [BC]. *[8 marks]*

4. *[Maximum mark: 20]*

The binary operation multiplication modulo 9, denoted by $\times_9$, is defined on the set $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

(a) Copy and complete the following Cayley table.

$\times_9$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	1	2	3	4	5	6	7	8
2	2	4	6	8	1	3	5	7
3								
4	4	8	3	7	2	6	1	5
5								
6	6	3	0	6	3	0	6	3
7								
8	8	7	6	5	4	3	2	1

[3 marks]

(b) Show that $\{S, \times_9\}$ is not a group. *[1 mark]*

(c) Prove that a group $\{G, \times_9\}$ can be formed by removing two elements from the set S. *[5 marks]*

(d) (i) Find the order of all the elements of G.

(ii) Write down all the proper subgroups of $\{G, \times_9\}$.

(iii) Determine the coset containing the element 5 for each of the subgroups in part (d)(ii). *[8 marks]*

(e) Solve the equation $4 \times_9 x \times_9 x = 1$. *[3 marks]*

5. *[Maximum mark: 24]*

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\sec x = x(\sec x - \tan x), \text{ where } y = 3 \text{ when } x = 0.$$

(a) Use Euler's method with a step length of 0.1 to find an approximate value for y when $x = 0.3$. *[5 marks]*

(b) (i) By differentiating the above differential equation, obtain an expression involving $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

(ii) Hence determine the Maclaurin series for y up to the term in x^2.

(iii) Use the result in part (b)(ii) to obtain an approximate value for y when $x = 0.3$. *[8 marks]*

(c) (i) Show that $\sec x + \tan x$ is an integrating factor for solving this differential equation.

(ii) Solve the differential equation, giving your answer in the form $y = f(x)$.

(iii) Hence determine which of the two approximate values for y when $x = 0.3$, obtained in parts (a) and (b), is closer to the true value. *[11 marks]*

6. *[Maximum mark: 25]*

(a) A connected planar graph has e edges, f faces and v vertices. Prove Euler's relation, that is $v + f = e + 2$. *[8 marks]*

(b) (i) A simple connected planar graph with v vertices, where $v \geq 3$, has no circuit of length 3. Deduce that $e \geq 2f$ and therefore that $e \leq 2v - 4$.

(ii) Hence show that $\kappa_{3,3}$ is non-planar. *[9 marks]*

(c) The graph P has the following adjacency table, defined for vertices A to H, where each element represents the number of edges between the respective vertices.

	A	B	C	D	E	F	G	H
A	0	1	1	0	0	0	0	0
B	1	0	0	1	0	0	1	0
C	1	0	0	1	0	0	0	0
D	0	1	1	0	0	1	0	0
E	0	0	0	0	0	0	1	1
F	0	0	0	1	0	0	0	1
G	0	1	0	0	1	0	0	0
H	0	0	0	0	1	1	0	0

(i) Show that P is bipartite.

(ii) Show that the complement of P is connected but not planar. *[8 marks]*

7. *[Maximum mark: 22]*

The random variable X has cumulative distribution function

$$F(x)=\begin{cases} 0 & x<0 \\ \left(\frac{x}{a}\right)^3 & 0\le x\le a \\ 1 & x>a \end{cases}$$

where a is an unknown parameter. You are given that the mean and variance of X are $\frac{3a}{4}$ and $\frac{3a^2}{80}$ respectively. To estimate the value of a, a random sample of n independent observations, $X_1, X_2, \ldots, X_n$, is taken from the distribution of X.

(a) (i) Find an expression for c in terms of n such that $U=c\sum_{i=1}^{n}X_i$ is an unbiased estimator for a.

(ii) Determine an expression for $\mathrm{Var}(U)$ in this case. *[4 marks]*

(b) Let Y denote the largest value of X in the random sample.

(i) Show that $\mathrm{P}(Y\le y)=\left(\frac{y}{a}\right)^{3n}$, $0\le y\le a$ and deduce an expression for the probability density function of Y.

(ii) Find $\mathrm{E}(Y)$.

(iii) Show that $\mathrm{Var}(Y)=\frac{3na^2}{(3n+2)(3n+1)^2}$.

(iv) Find an expression for d in terms of n such that $V=dY$ is an unbiased estimator for a.

(v) Determine an expression for $\mathrm{Var}(V)$ in this case. *[16 marks]*

(c) Show that $\frac{\mathrm{Var}(U)}{\mathrm{Var}(V)}=\frac{3n+2}{5}$ and hence state, with a reason, which of U or V is the more efficient estimator for a. *[2 marks]*

8. *[Maximum mark: 25]*

(a) Given that the elements of a 2×2 symmetric matrix are real, show that

(i) the eigenvalues are real;

(ii) the eigenvectors are orthogonal if the eigenvalues are distinct. *[11 marks]*

(b) The matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A}=\begin{pmatrix}11 & \sqrt{3}\\ \sqrt{3} & 9\end{pmatrix}.$$

Find the eigenvalues and eigenvectors of $\boldsymbol{A}$. *[7 marks]*

(c) The ellipse E has equation $\boldsymbol{X}^T\boldsymbol{A}\boldsymbol{X}=24$ where $\boldsymbol{X}=\begin{pmatrix}x\\ y\end{pmatrix}$ and $\boldsymbol{A}$ is as defined in part (b).

(i) Show that E can be rotated about the origin onto the ellipse E' having equation $2x^2+3y^2=6$.

(ii) Find the acute angle through which E has to be rotated to coincide with E'. *[7 marks]*

MARKSCHEME

SPECIMEN

FURTHER MATHEMATICS

Higher Level

Paper 2

14 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) are often dependent on the preceding ***M*** mark.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.* do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled ***(d)*** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates may not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

The method of dealing with accuracy errors on a whole paper basis by means of the Accuracy Penalty ***(AP)*** *no longer applies.*

Instructions to examiners about such numerical issues will be provided on a question by question basis within the framework of mathematical correctness, numerical understanding and contextual appropriateness.

The rubric on the front page of each question paper is given for the guidance of candidates. The markscheme (MS) may contain instructions to examiners in the form of "Accept answers which round to n significant figures (sf)". Where candidates state answers, required by the question, to fewer than n sf, award ***A0****. Some intermediate numerical answers may be required by the MS but not by the question. In these cases only award the mark(s) if the candidate states the answer exactly or to at least 2sf.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (e.g. TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) $\frac{x_1}{x_1} = \frac{y_1}{y_1} \Rightarrow (x_1, y_1) R (x_1, y_1)$ so R is reflexive ***R1***

$(x_1, y_1) R (x_2, y_2) \Rightarrow \frac{x_1}{x_2} = \frac{y_2}{y_1} \Rightarrow \frac{x_2}{x_1} = \frac{y_1}{y_2} \Rightarrow (x_2, y_2) R (x_1, y_1)$ ***M1A1***

so R is symmetric

$(x_1, y_1) R (x_2, y_2)$ and $(x_2, y_2) R (x_3, y_3) \Rightarrow \frac{x_1}{x_2} = \frac{y_2}{y_1}$ and $\frac{x_2}{x_3} = \frac{y_3}{y_2}$ ***M1***

multiplying the two equations, ***M1***

$\Rightarrow \frac{x_1}{x_3} = \frac{y_3}{y_1} \Rightarrow (x_1, y_1) R (x_3, y_3)$ so R is transitive ***A1***

thus R is an equivalence relation ***AG***

[6 marks]

(b) $(x, y) R (x_1, y_1) \Rightarrow \frac{x}{x_1} = \frac{y_1}{y} \Rightarrow xy = x_1 y_1$ ***(M1)***

the equivalence class is therefore $\{(x, y) \mid xy = x_1 y_1\}$ ***A1***

geometrically, the equivalence class is (one branch of) a (rectangular) hyperbola ***A1***

[3 marks]

Total [9 marks]

2. (a) (i) from the GDC,

unbiased estimate for $\mu = 96.3$ ***A1***

unbiased estimate for $\sigma^2 = 8.028\ldots^2 = 64.5$ ***(M1)A1***

(ii) 95% confidence interval is $[90.6, 102]$ ***A1A1***

Note: Accept 102.0 as the upper limit.

[5 marks]

(b) (i) $H_0 : \mu = 100$; $H_1 : \mu < 100$ ***A1***

(ii) $\bar{x} = 98.31$, $S_{n-1} = 7.8446\ldots$ ***(A1)***

p-value $= 0.0168$ ***A1***

(iii) the farm manager's claim is accepted because $0.0168 > 0.01$ ***A1R1***

[5 marks]

Total [10 marks]

3. (a) (i) CEFB is cyclic because $\text{B}\hat{\text{E}}\text{C} = \text{B}\hat{\text{F}}\text{C} = 90^\circ$ ***R1***
([BC] is actually the diameter)

(ii) consider the triangles CHE, CPE ***M1***
[CE] is common ***A1***
$\text{H}\hat{\text{E}}\text{C} = \text{P}\hat{\text{E}}\text{C} = 90^\circ$ ***A1***
$\text{P}\hat{\text{C}}\text{E} = \text{P}\hat{\text{B}}\text{A}$ (subtended by chord [AP]) ***A1***
$\text{P}\hat{\text{B}}\text{A} = \text{F}\hat{\text{C}}\text{E}$ (subtended by chord [FE]) ***A1***
triangles CHE and CPE are congruent ***A1***
therefore $\text{HE} = \text{EP}$ ***AG***

[7 marks]

(b) (i) EAFH is a cyclic quad because $\text{A}\hat{\text{E}}\text{B} = \text{C}\hat{\text{F}}\text{A} = 90^\circ$ ***M1***
$\text{C}\hat{\text{A}}\text{D} = \text{E}\hat{\text{F}}\text{H}$ subtended by the chord [HE] ***R1AG***
CEFB is a cyclic quad from part (a) ***M1***
$\text{E}\hat{\text{F}}\text{H} = \text{E}\hat{\text{B}}\text{C}$ subtended by the chord [EC] ***R1AG***

(ii) $\text{A}\hat{\text{D}}\text{C} = 180^\circ - \text{C}\hat{\text{A}}\text{D} - \text{D}\hat{\text{C}}\text{A}$ ***M1***
$= 180^\circ - \text{C}\hat{\text{A}}\text{D} - (90 - \text{E}\hat{\text{B}}\text{C})$ ***A1***
$= 90^\circ - \text{C}\hat{\text{A}}\text{D} + \text{E}\hat{\text{B}}\text{C}$ ***A1***
$= 90^\circ$ ***A1***
hence [AD] is perpendicular to [BC] ***AG***

[8 marks]

Total [15 marks]

4. (a)

$\times_9$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	1	2	3	4	5	6	7	8
2	2	4	6	8	1	3	5	7
3	*3*	*6*	*0*	*3*	*6*	*0*	*3*	*6*
4	4	8	3	7	2	6	1	5
5	*5*	*1*	*6*	*2*	*7*	*3*	*8*	*4*
6	6	3	0	6	3	0	6	3
7	*7*	*5*	*3*	*1*	*8*	*6*	*4*	*2*
8	8	7	6	5	4	3	2	1

A3

Note: Award ***A2*** if one error, ***A1*** if two errors and ***A0*** if three or more errors.

[3 marks]

(b) any valid reason, ***R1***

e.g. not closed

3 or 6 has no inverse,

it is not a Latin square

[1 mark]

(c) remove 3 and 6 ***A1***

for the remaining elements,

the table is closed ***R1***

associative because multiplication is associative ***R1***

the identity is 1 ***A1***

every element has an inverse, $(2, 5)$ and $(4, 7)$ are

inverse pairs and 8 (and 1) are self-inverse ***A1***

thus it is a group ***AG***

[5 marks]

(d) (i) the orders are

element	order
1	1
2	6
4	3
5	6
7	3
8	2

A3

Note: Award ***A2*** if one error, ***A1*** if two errors and ***A0*** if three or more errors.

(ii) the proper subgroups are

$\{1, 8\}$ ***A1***

$\{1, 4, 7\}$ ***A1***

Note: Do not penalize inclusion of $\{1\}$.

(iii) the cosets are $\{5, 4\}$ ***(M1)A1***

$\{5, 2, 8\}$ ***A1***

[8 marks]

continued …

Question 4 continued

(e) $x \times_9 x = 7$ ***(A1)***

$x = 4, 5$ ***A1A1***

[3 marks]

Total [20 marks]

5. (a)

x	y	dy/dx	0.1 dy/dx	
0	3	–3	–0.3	***(M1)***
0.1	2.7	–2.623087855	–0.2623087855	***(A1)***
0.2	2.437691214	–2.323745276	–0.2323745276	***(A1)***
0.3	2.205316686			***(A1)***

Note: The ***A1*** marks above are for correct entries in the y column.

$y(0.3) \approx 2.21$ ***A1***

[5 marks]

(b) (i) use of product rule on either side ***M1***

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \sec x\frac{\mathrm{d}y}{\mathrm{d}x} + y\sec x\tan x = \sec x - \tan x + x(\sec x\tan x - \sec^2 x)$$ ***A1A1***

(ii) $y(0) = 3$

$y'(0) = -3,\ y''(0) = 4$ ***A1A1***

the quadratic approximation is

$$y = \left(y(0) + xy'(0) + \frac{x^2y''(0)}{2} = \right)3 - 3x + 2x^2$$ ***(M1)A1***

(iii) using this approximation, $y(0.3) \approx 2.28$ ***A1***

[8 marks]

continued …

Question 5 continued

(c) (i) **EITHER**

$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x + \tan x) = \sec x \tan x + \sec^2 x$ ***A1***

$\sec x(\sec x + \tan x) = \sec^2 x + \sec x \tan x$ ***A1***

as these two expressions are the same, this is an integrating factor ***R1AG***

OR

$(\sec x + \tan x)\left(\frac{\mathrm{d}y}{\mathrm{d}x} + y \sec x\right) = (\sec x + \tan x)\, x(\sec x - \tan x)$ ***M1***

Note: RHS does not need to be shown.

$\text{LHS} = \frac{\mathrm{d}y}{\mathrm{d}x}(\sec x + \tan x) + y(\sec x \tan x + \sec^2 x)$ ***A1***

$= \frac{\mathrm{d}}{\mathrm{d}x} y(\sec x + \tan x)$ ***A1***

making LHS an exact derivative

OR

integrating factor $= \mathrm{e}^{\int \sec x \,\mathrm{d}x}$ ***M1***

since $\frac{\mathrm{d}}{\mathrm{d}x}\ln(\sec x + \tan x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x$ ***M1A1***

integrating factor $= \mathrm{e}^{\ln(\sec x + \tan x)} = \sec x + \tan x$ ***AG***

(ii) $\frac{\mathrm{d}}{\mathrm{d}x}(y[\sec x + \tan x]) = x(\sec^2 x - \tan^2 x) = x$ ***M1A1***

$y(\sec x + \tan x) = \frac{x^2}{2} + c$ ***A1***

$x = 0,\ y = 3 \Rightarrow c = 3$ ***M1A1***

$y = \frac{x^2 + 6}{2(\sec x + \tan x)}$ ***A1***

(iii) when $x = 0.3,\ y = 2.245\ldots$ ***A1***

the closer approximation is obtained by using the series in part (b) ***R1***

[11 marks]

Total [24 marks]

6. (a) consider the basic graph with just 1 vertex for which $v=1$, $f=1$ and $e=0$
for this case, $v+f=e+2=2$ so the result is true here ***M1A1***

Note: Allow solutions which begin with a graph containing 2 vertices and an edge joining them for which $v=2$, $f=1$ and $e=1$.

a graph can be extended as follows – there are three cases to consider
I – an extra edge is added joining two distinct existing vertices ***R1***
II – an extra edge is added joining an existing vertex to itself,
forming a loop ***R1***
in each case *v* remains the same and *f* and *e* each increase by 1
both sides of the equation increase by 1 and the equation still holds ***R1***
III – an extra vertex is added together with an edge joining this new vertex
to an existing vertex (which is necessary to keep the graph connected) ***R1***
in this case, f remains the same and *v* and *e* each increase by 1
both sides of the equation increase by 1 and the equation still holds ***R1***
any graph can be constructed from the basic graph by combining these
operations, all of which result in Euler's relation remaining valid ***R1***

[8 marks]

(b) (i) since the graph is simple there are no loops and no multiple edges
and thus no circuits of length 1 or 2 ***R1***
as we are told that there are no circuits of length 3, any face
must be surrounded by at least 4 edges ***R1***
since every edge is adjacent to 2 faces, ***R1***
$2e=\sum$ (degrees of faces) $\geq 4f$, ***A1***
it follows that $e \geq 2f$ ***AG***

using Euler's relation with $f \leq \frac{e}{2}$, ***M1***

$f=e-v+2 \leq \frac{e}{2}$ ***A1***

giving $e \leq 2v-4$ ***AG***

(ii) $\kappa_{3,3}$ is simple and since it is bipartite it has no cycles of length 3 ***R1***
for $\kappa_{3,3}$, $v=6$ and $e=9$ ***A1***
$2v-4=8$ so that the above inequality is not satisfied ***R1***
it follows that $\kappa_{3,3}$ is not planar ***AG***

[9 marks]

continued …

Question 6 continued

(c) (i) attempt to find disjoint sets of vertices ***(M1)***
disjoint sets are $\{A, D, G, H\}$ and $\{B, C, E, F\}$ ***A1***

Note: Accept graph with vertices coloured, or otherwise annotated.

(ii) let P' denote the complement of P
in P', A is connected to D, E, F, G and H : B and C are connected to E
therefore A is connected to all other vertices so P' is connected ***M1A1***
a complete graph with 8 vertices has 28 edges ***A1***
since P has 9 edges, P' has 19 edges ***A1***
consider $e \le 3v - 6$ (the condition for a planar graph) ***M1***
for P', $e = 19$; $3v - 6 = 18$ so the condition is not satisfied ***A1***
therefore P' is not planar ***AG***

[8 marks]

Total [25 marks]

7. (a) (i) $\mathrm{E}(U) = c \times n \times \dfrac{3a}{4} = a \Rightarrow c = \dfrac{4}{3n}$ ***M1A1***

(ii) $\mathrm{Var}(U) = \dfrac{16}{9n^2} \times n \times \dfrac{3a^2}{80} = \dfrac{a^2}{15n}$ ***M1A1***

[4 marks]

(b) (i) $P(Y \le y) = P(\text{all } X\text{s } \le y)$ ***M1***

$= [P(X \le y)]^n$ ***(A1)***

$= \left(\left(\dfrac{y}{a}\right)^3\right)^n$ ***(A1)***

Note: Only one of the two ***A1*** marks above may be implied.

$= \left(\dfrac{y}{a}\right)^{3n}$ ***AG***

$g(y) = \dfrac{\mathrm{d}}{\mathrm{d}y}\left(\dfrac{y}{a}\right)^{3n} = \dfrac{3ny^{3n-1}}{a^{3n}},\ (0 < y < a)$ ***M1A1***

$\big(g(y) = 0 \text{ otherwise}\big)$

(ii) $\mathrm{E}(Y) = \displaystyle\int_0^a \frac{3ny^{3n}}{a^{3n}}\,\mathrm{d}y$ ***M1***

$= \left[\dfrac{3ny^{3n+1}}{(3n+1)a^{3n}}\right]_0^a$ ***A1***

$= \dfrac{3na}{3n+1}$ ***A1***

(iii) $\mathrm{Var}(Y) = \displaystyle\int_0^a \frac{3ny^{3n+1}}{a^{3n}}\,\mathrm{d}y - \left(\frac{3na}{3n+1}\right)^2$ ***M1***

$= \left[\dfrac{3ny^{3n+2}}{(3n+2)a^{3n}}\right]_0^a - \left(\dfrac{3na}{3n+1}\right)^2$ ***A1***

$= \dfrac{3na^2}{3n+2} - \dfrac{9n^2a^2}{(3n+1)^2}$ ***M1***

$= \dfrac{3na^2(9n^2+6n+1) - 9n^2a^2(3n+2)}{(3n+2)(3n+1)^2}$ ***A1***

$= \dfrac{3na^2}{(3n+2)(3n+1)^2}$ ***AG***

(iv) $\mathrm{E}(V) = d \times \dfrac{3na}{3n+1} = a \Rightarrow d = \dfrac{3n+1}{3n}$ ***M1A1***

continued …

Question 7 continued

(v) $\text{Var}(V) = \left(\frac{3n+1}{3n}\right)^2 \times \frac{3na^2}{(3n+2)(3n+1)^2}$ ***M1***

$= \frac{a^2}{3n(3n+2)}$ ***A1***

[16 marks]

(c) $\frac{\text{Var}(U)}{\text{Var}(V)} = \frac{\frac{a^2}{15n}}{\frac{a^2}{3n(3n+2)}}$ ***A1***

$= \frac{3n+2}{5}$ ***AG***

V is the more efficient estimator because $3n+2>5$ (for $n>1$) ***R1***

[2 marks]

Total [22 marks]

8. (a) (i) let $\boldsymbol{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ ***(M1)***

the eigenvalues satisfy

$\det(\boldsymbol{M} - \lambda \boldsymbol{I}) = 0$ ***(M1)***

$(a-\lambda)(c-\lambda) - b^2 = 0$ ***(A1)***

$\lambda^2 - \lambda(a+c) + ac - b^2 = 0$ ***A1***

discriminant $= (a+c)^2 - 4(ac-b^2)$ ***M1***

$= (a-c)^2 + 4b^2 \ge 0$ ***A1***

this shows that the eigenvalues are real ***AG***

(ii) let the distinct eigenvalues be λ_1, λ_2 with eigenvectors $\boldsymbol{X}_1$, $\boldsymbol{X}_2$

then

$\lambda_1 \boldsymbol{X}_1 = \boldsymbol{M}\boldsymbol{X}_1$ and $\lambda_2 \boldsymbol{X}_2 = \boldsymbol{M}\boldsymbol{X}_2$ ***M1***

transpose the first equation and postmultiply by $\boldsymbol{X}_2$ to give

$\lambda_1 \boldsymbol{X}_1^T \boldsymbol{X}_2 = \boldsymbol{X}_1^T \boldsymbol{M}\boldsymbol{X}_2$ ***A1***

premultiply the second equation by $\boldsymbol{X}_1^T$

$\lambda_2 \boldsymbol{X}_1^T \boldsymbol{X}_2 = \boldsymbol{X}_1^T \boldsymbol{M}\boldsymbol{X}_2$ ***A1***

it follows that

$(\lambda_1 - \lambda_2)\boldsymbol{X}_1^T \boldsymbol{X}_2 = \boldsymbol{0}$ ***A1***

since $\lambda_1 \ne \lambda_2$, it follows that $\boldsymbol{X}_1^T \boldsymbol{X}_2 = \boldsymbol{0}$ so that the eigenvectors are orthogonal ***R1***

[11 marks]

continued …

Question 8 continued

(b) the eigenvalues satisfy

$$\begin{vmatrix} 11-\lambda & \sqrt{3} \\ \sqrt{3} & 9-\lambda \end{vmatrix} = 0$$ ***M1A1***

$\lambda^2 - 20\lambda + 96 = 0$ ***A1***

$\lambda = 8, 12$ ***A1***

first eigenvector satisfies

$$\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ ***M1***

$$\begin{pmatrix} x \\ y \end{pmatrix} = (\text{any multiple of})\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$ ***A1***

second eigenvector satisfies

$$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = (\text{any multiple of})\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$ ***A1***

[7 marks]

(c) (i) consider the rotation in which (x, y) is transformed onto (x', y') defined by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \text{ so that } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$ ***M1A1***

the ellipse *E* becomes

$$(x' \quad y')\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 11 & \sqrt{3} \\ \sqrt{3} & 9 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 24$$ ***M1A1***

$$(x' \quad y')\begin{pmatrix} 8 & 0 \\ 0 & 12 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 24$$ ***A1***

$2(x')^2 + 3(y')^2 = 6$ ***AG***

(ii) the angle of rotation is given by $\cos\theta = \frac{1}{2}, \sin\theta = \frac{\sqrt{3}}{2}$ ***M1***

since a rotational matrix has the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

so $\theta = 60^\circ$ (anticlockwise) ***A1***

[7 marks]

Total [25 marks]