Mathematics
Higher level
Paper 3 – sets, relations and groups

Friday 18 November 2016 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

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Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 13]

Let $\{G, \circ\}$ be the group of all permutations of $1, 2, 3, 4, 5, 6$ under the operation of composition of permutations.

(a) (i) Write the permutation $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 6 & 2 & 1 & 5 \end{pmatrix}$ as a composition of disjoint cycles.

(ii) State the order of α. [3]

(b) (i) Write the permutation $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 5 & 1 & 2 \end{pmatrix}$ as a composition of disjoint cycles.

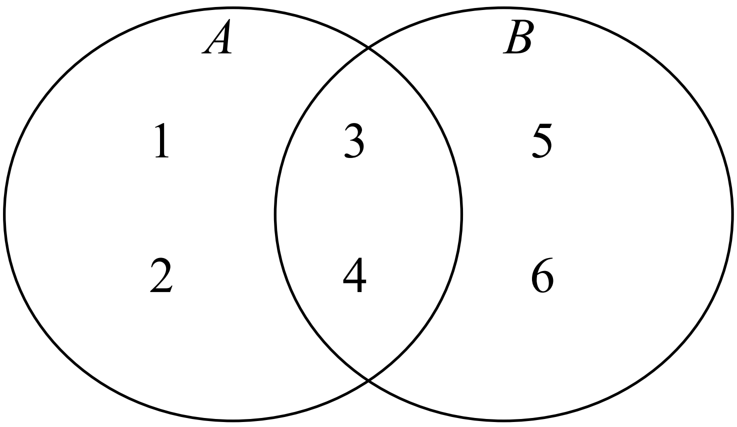
(ii) State the order of β. [2]

(c) Write the permutation $\alpha \circ \beta$ as a composition of disjoint cycles. [2]

(d) Write the permutation $\beta \circ \alpha$ as a composition of disjoint cycles. [2]

(e) State the order of $\{G, \circ\}$. [2]

Consider the following Venn diagram, where $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$.



(f) Find the number of permutations in $\{G, \circ\}$ which will result in A, B and $A \cap B$ remaining unchanged. [2]

2. [Maximum mark: 21]

(a) Let A be the set $\{x \mid x \in \mathbb{R}, x \neq 0\}$. Let B be the set $\{x \mid x \in]-1, +1[, x \neq 0\}$.

A function $f: A \to B$ is defined by $f(x) = \frac{2}{\pi}\arctan(x)$.

(i) Sketch the graph of $y = f(x)$ and hence justify whether or not f is a bijection.

(ii) Show that A is a group under the binary operation of multiplication.

(iii) Give a reason why B is not a group under the binary operation of multiplication.

(iv) Find an example to show that $f(a \times b) = f(a) \times f(b)$ is not satisfied for all $a, b \in A$. [13]

(b) Let D be the set $\{x \mid x \in \mathbb{R}, x > 0\}$.

A function $g: \mathbb{R} \to D$ is defined by $g(x) = \mathrm{e}^x$.

(i) Sketch the graph of $y = g(x)$ and hence justify whether or not g is a bijection.

(ii) Show that $g(a + b) = g(a) \times g(b)$ for all $a, b \in \mathbb{R}$.

(iii) Given that $\{\mathbb{R}, +\}$ and $\{D, \times\}$ are both groups, explain whether or not they are isomorphic. [8]

3. [Maximum mark: 15]

An Abelian group, $\{G, *\}$, has 12 different elements which are of the form $a^i * b^j$ where $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, 3\}$. The elements a and b satisfy $a^4 = e$ and $b^3 = e$ where e is the identity.

(a) State the possible orders of an element of $\{G, *\}$ and for each order give an example of an element of that order. [8]

Let $\{H, *\}$ be the proper subgroup of $\{G, *\}$ having the maximum possible order.

(b) (i) State a generator for $\{H, *\}$.

(ii) Write down the elements of $\{H, *\}$.

(iii) Write down the elements of the coset of H containing a. [7]

4. [Maximum mark: 11]

A relation S is defined on $\mathbb{R}$ by aSb if and only if $ab > 0$.

(a) Show that S is

(i) not reflexive;

(ii) symmetric;

(iii) transitive. [4]

A relation R is defined on a non-empty set A. R is symmetric and transitive but not reflexive.

(b) Explain why there exists an element $a \in A$ that is not related to itself. [1]

(c) Hence prove that there is at least one element of A that is not related to any other element of A. [6]