22147102

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

FURTHER MATHEMATICS
HIGHER LEVEL
PAPER 2

Thursday 22 May 2014 (morning)

2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[150 marks]*.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 9]*

The random variable X has the binomial distribution $\mathrm{B}(n,\ p)$, where $n > 1$.

Show that

(a) $\dfrac{X}{n}$ is an unbiased estimator for p; *[2]*

(b) $\left(\dfrac{X}{n}\right)^2$ is **not** an unbiased estimator for p^2; *[4]*

(c) $\dfrac{X(X-1)}{n(n-1)}$ is an unbiased estimator for p^2. *[3]*

2. *[Maximum mark: 18]*

(a) The set S contains the eighth roots of unity given by $\left\{\text{cis}\left(\frac{n\pi}{4}\right), n \in \mathbb{N}, 0 \le n \le 7\right\}$.

(i) Show that $\{S, \times\}$ is a group where $\times$ denotes multiplication of complex numbers.

(ii) Giving a reason, state whether or not $\{S, \times\}$ is cyclic. *[7]*

(b) The group $\{G, \times_{20}\}$ is defined on the set $\{1, 3, 7, 9, 11, 13, 17, 19\}$ where $\times_{20}$ denotes multiplication modulo 20.

(i) Copy and complete the following Cayley table for $\{G, \times_{20}\}$.

	1	**3**	**7**	**9**	**11**	**13**	**17**	**19**
1	1	3	7	9	11	13	17	19
3	3	9	1	7	13	19	11	17
7	7	1	9	3	17	11	19	13
9	9	7	3					
11	11	13	17					
13	13	19	11					
17	17	11	19					
19	19	17	13					

(ii) Determine the order of each element of $\{G, \times_{20}\}$.

(iii) Giving a reason, state whether or not $\{S, \times\}$ and $\{G, \times_{20}\}$ are isomorphic.

(iv) Find a cyclic subgroup of $\{G, \times_{20}\}$ of order 4 and state all its generators.

(v) Find a non-cyclic subgroup of $\{G, \times_{20}\}$ of order 4. *[11]*

3. *[Maximum mark: 19]*

The vertices and weights of the graph G are given in the following table.

Vertices	**A**	**B**	**C**	**D**	**E**	**F**
A	–	18	19	17	20	21
B	18	–	14	21	12	10
C	19	14	–	20	15	20
D	17	21	20	–	16	22
E	20	12	15	16	–	13
F	21	10	20	22	13	–

(a) (i) Use Kruskal's algorithm to find the minimum spanning tree for G, indicating clearly the order in which the edges are included.

(ii) Draw the minimum spanning tree for G. *[4]*

(b) Consider the travelling salesman problem for G.

(i) An upper bound for the problem can be found by doubling the weight of the minimum spanning tree. Use this method to find an upper bound.

(ii) Starting at A, use the nearest neighbour algorithm to find another upper bound.

(iii) By first removing A, use the deleted vertex algorithm to find a lower bound for the problem. *[10]*

(c) The travelling salesman problem is now modified so that starting at A, the vertices B and C have to be visited first in that order, then D, E, F in any order before returning to A.

(i) Solve this modified problem.

(ii) Comment whether or not your answer has any effect on the upper bound to the problem considered in (b). *[5]*

4. *[Maximum mark: 18]*

The matrix $\boldsymbol{A}$ is given by $\boldsymbol{A}=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 8 & 11 & 8 \\ 1 & 3 & 4 & \lambda \\ \lambda & 5 & 7 & 6 \end{pmatrix}$.

(a) Given that $\lambda = 2$, $\boldsymbol{B}=\begin{pmatrix} 2 \\ 4 \\ \mu \\ 3 \end{pmatrix}$ and $\boldsymbol{X}=\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$,

(i) find the value of μ for which the equations defined by $\boldsymbol{AX}=\boldsymbol{B}$ are consistent and solve the equations in this case;

(ii) define the rank of a matrix and state the rank of $\boldsymbol{A}$. *[10]*

(b) Given that $\lambda = 1$,

(i) show that the four column vectors in $\boldsymbol{A}$ form a basis for the space of four-dimensional column vectors;

(ii) express the vector $\begin{pmatrix} 6 \\ 28 \\ 12 \\ 15 \end{pmatrix}$ as a linear combination of these basis vectors. *[8]*

5. *[Maximum mark: 18]*

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = 2\cos^4 x$ given that $y=1$ when $x=0$.

(a) Solve the differential equation, giving your answer in the form $y = f(x)$. *[9]*

(b) (i) By differentiating both sides of the differential equation, show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = -10\sin x\cos^3 x.$$

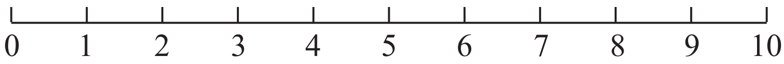
(ii) Hence find the first four terms of the Maclaurin series for y. *[9]*

6. *[Maximum mark: 13]*

(a) Consider the recurrence relation $au_{n+1} + bu_n + cu_{n-1} = 0$.

Show that $u_n = A\lambda^n + B\mu^n$ satisfies this relation where A, B are arbitrary constants and λ, μ are the roots of the equation $ax^2 + bx + c = 0$. *[3]*

(b)



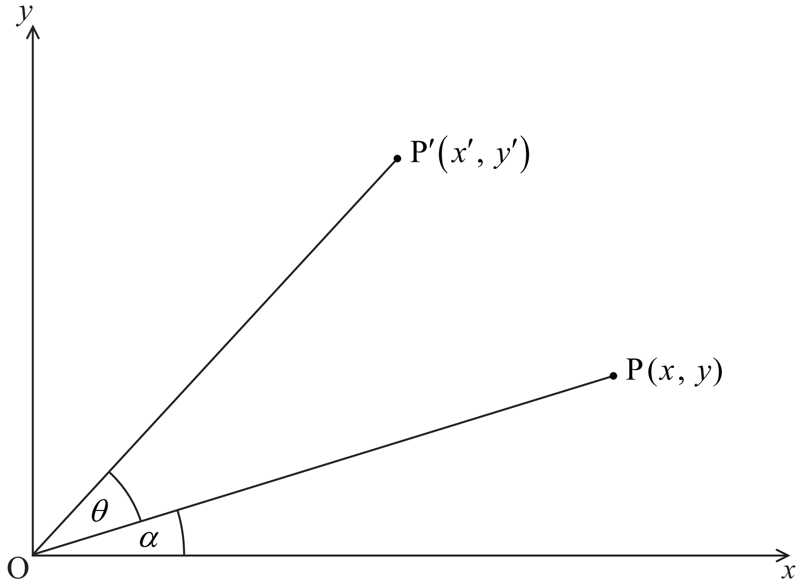
A particle P executes a random walk on the line above such that when it is at point $n\left(1 \le n \le 9,\ n \in \mathbb{Z}^+\right)$ it has a probability 0.4 of moving to $n+1$ and a probability 0.6 of moving to $n-1$. The walk terminates as soon as P reaches either 0 or 10. Let p_n denote the probability that the walk terminates at 0 starting from n.

(i) Show that $2p_{n+1} - 5p_n + 3p_{n-1} = 0$.

(ii) By solving this recurrence relation subject to the boundary conditions $p_0 = 1$, $p_{10} = 0$ show that $p_n = \dfrac{1.5^{10} - 1.5^n}{1.5^{10} - 1}$. *[10]*

7. *[Maximum mark: 14]*

(a)



The diagram above shows the points $\mathrm{P}(x, y)$ and $\mathrm{P}'(x', y')$ which are equidistant from the origin O. The line (OP) is inclined at an angle α to the x-axis and $\mathrm{P\hat{O}P'} = \theta$.

(i) By first noting that $\mathrm{OP} = x\sec\alpha$, show that $x' = x\cos\theta - y\sin\theta$ and find a similar expression for y'.

(ii) Hence write down the 2×2 matrix which represents the anticlockwise rotation about O which takes P to P'. *[7]*

(b) The ellipse E has equation $5x^2 + 5y^2 - 6xy = 8$.

(i) Show that if E is rotated **clockwise** about the origin through 45°, its equation becomes $\frac{x^2}{4} + y^2 = 1$.

(ii) Hence determine the coordinates of the foci of E. *[7]*

8. *[Maximum mark: 27]*

(a) (i) Using l'Hôpital's rule, show that

$$\lim_{x\to\infty}\frac{x^n}{e^{\lambda x}}=0;\ n\in\mathbb{Z}^+,\ \lambda\in\mathbb{R}^+$$

(ii) Using mathematical induction on n, prove that

$$\int_0^\infty x^n\,e^{-\lambda x}\,dx=\frac{n!}{\lambda^{n+1}};\ n\in\mathbb{N},\ \lambda\in\mathbb{R}^+$$ *[13]*

(b) The random variable X has probability density function

$$f(x)=\begin{cases}\dfrac{\lambda^{n+1}x^n e^{-\lambda x}}{n!} & x\ge 0,\ n\in\mathbb{Z}^+,\ \lambda\in\mathbb{R}^+\\ \text{otherwise} & \end{cases}$$

Giving your answers in terms of n and λ, determine

(i) $E(X)$;

(ii) the mode of X. *[6]*

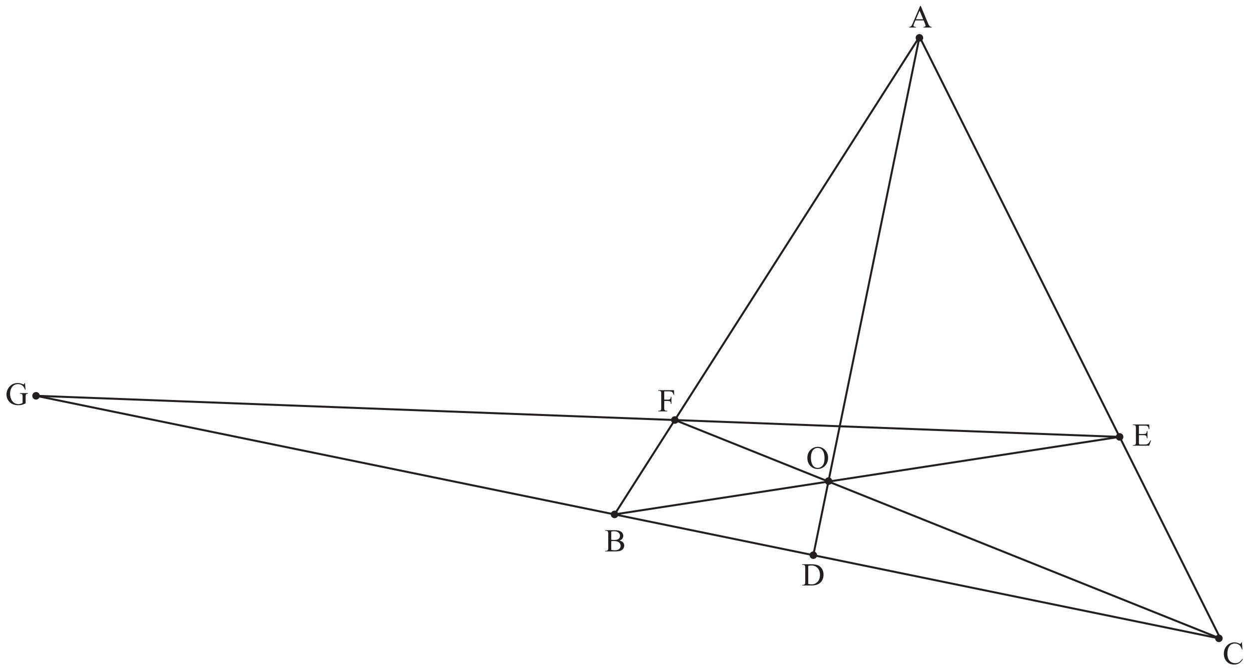
(c) Customers arrive at a shop such that the number of arrivals in any interval of duration d hours follows a Poisson distribution with mean $8d$. The third customer on a particular day arrives T hours after the shop opens.

(i) Show that $P(T>t)=e^{-8t}\left(1+8t+32t^2\right)$.

(ii) Find an expression for the probability density function of T.

(iii) Deduce the mean and the mode of T. *[8]*

9. *[Maximum mark: 14]*



The diagram above shows a point O inside a triangle ABC. The lines (AO), (BO), (CO) meet the lines (BC), (CA), (AB) at the points D, E, F respectively. The lines (EF), (BC) meet at the point G.

(a) Show that, with the usual convention for the signs of lengths in a triangle, $\frac{\text{BD}}{\text{DC}} = -\frac{\text{BG}}{\text{GC}}$. *[6]*

(b) The lines (FD), (CA) meet at the point H and the lines (DE), (AB) meet at the point I. Show that the points G, H, I are collinear. *[8]*