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MATHEMATICS
HIGHER LEVEL
PAPER 3 – STATISTICS AND PROBABILITY

Thursday 13 November 2014 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 9]*

A random variable X has probability density function

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \le x < 1 \\ \frac{1}{4} & 1 \le x < 3 \\ 0 & x \ge 3 \end{cases}.$$

(a) Sketch the graph of $y = f(x)$. *[1]*

(b) Find the cumulative distribution function for X. *[5]*

(c) Find the interquartile range for X. *[3]*

2. *[Maximum mark: 9]*

Eric plays a game at a fairground in which he throws darts at a target. Each time he throws a dart, the probability of hitting the target is 0.2. He is allowed to throw as many darts as he likes, but it costs him \$1 a throw. If he hits the target a total of three times he wins \$10.

(a) Find the probability he has his third success of hitting the target on his sixth throw. *[3]*

(b) (i) Find the expected number of throws required for Eric to hit the target three times.

(ii) Write down his expected profit or loss if he plays until he wins the \$10. *[3]*

(c) If he has just \$8, find the probability he will lose all his money before he hits the target three times. *[3]*

3. *[Maximum mark: 11]*

(a) If X and Y are two random variables such that $\mathrm{E}(X) = \mu_X$ and $\mathrm{E}(Y) = \mu_Y$ then $\mathrm{Cov}(X, Y) = \mathrm{E}\big((X - \mu_X)(Y - \mu_Y)\big)$.

Prove that if X and Y are independent then $\mathrm{Cov}(X, Y) = 0$. *[3]*

(b) In a particular company, it is claimed that the distance travelled by employees to work is independent of their salary. To test this, 20 randomly selected employees are asked about the distance they travel to work and the size of their salaries. It is found that the product moment correlation coefficient, r, for the sample is -0.35.

You may assume that both salary and distance travelled to work follow normal distributions.

Perform a one-tailed test at the $5\,\%$ significance level to test whether or not the distance travelled to work and the salaries of the employees are independent. *[8]*

4. *[Maximum mark: 21]*

If X is a random variable that follows a Poisson distribution with mean $\lambda > 0$ then the probability generating function of X is $G(t) = e^{\lambda(t-1)}$.

(a) (i) Prove that $\mathrm{E}(X) = \lambda$.

(ii) Prove that $\mathrm{Var}(X) = \lambda$. *[6]*

Y is a random variable, independent of X, that also follows a Poisson distribution with mean λ.

(b) If $S = 2X - Y$ find

(i) $\mathrm{E}(S)$;

(ii) $\mathrm{Var}(S)$. *[3]*

Let $T = \dfrac{X}{2} + \dfrac{Y}{2}$.

(c) (i) Show that T is an unbiased estimator for λ.

(ii) Show that T is a more efficient unbiased estimator of λ than S. *[3]*

(d) Could either S or T model a Poisson distribution? Justify your answer. *[1]*

(e) By consideration of the probability generating function, $G_{X+Y}(t)$, of $X+Y$, prove that $X+Y$ follows a Poisson distribution with mean 2λ. *[3]*

(f) Find

(i) $G_{X+Y}(1)$;

(ii) $G_{X+Y}(-1)$. *[2]*

(g) Hence find the probability that $X+Y$ is an even number. *[3]*

5. *[Maximum mark: 10]*

Two species of plant, A and B, are identical in appearance though it is known that the mean length of leaves from a plant of species A is 5.2 cm, whereas the mean length of leaves from a plant of species B is 4.6 cm. Both lengths can be modelled by normal distributions with standard deviation 1.2 cm.

In order to test whether a particular plant is from species A or species B, 16 leaves are collected at random from the plant. The length, x, of each leaf is measured and the mean length evaluated. A one-tailed test of the sample mean, $\bar{X}$, is then performed at the 5 % level, with the hypotheses: $H_0 : \mu = 5.2$ and $H_1 : \mu < 5.2$.

(a) Find the critical region for this test. *[3]*

(b) Find the probability of a Type II error if the leaves are in fact from a plant of species B. *[2]*

It is now known that in the area in which the plant was found 90 % of all the plants are of species A and 10 % are of species B.

(c) Find the probability that $\bar{X}$ will fall within the critical region of the test. *[2]*

(d) If, having done the test, the sample mean is found to lie within the critical region, find the probability that the leaves came from a plant of species A. *[3]*