22097205

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 1

Thursday 7 May 2009 (afternoon)

2 hours

Candidate session number

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INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 5]*

When the function $q(x) = x^3 + kx^2 - 7x + 3$ is divided by $(x+1)$ the remainder is seven times the remainder that is found when the function is divided by $(x+2)$.

Find the value of k.

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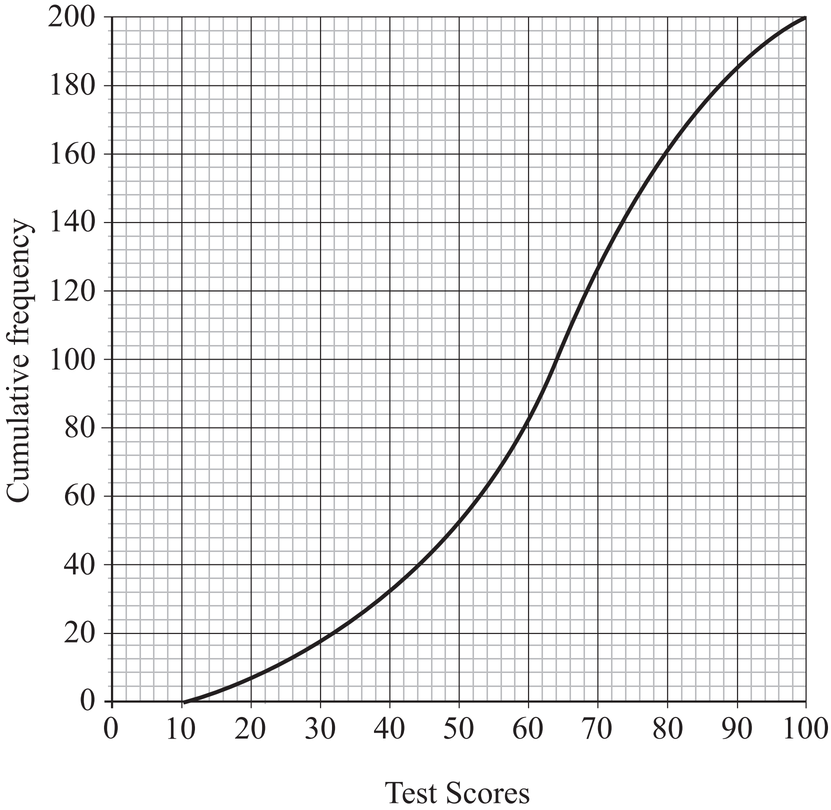
2. *[Maximum mark: 5]*

The test scores of a group of students are shown on the cumulative frequency graph below.



(a) Estimate the median test score. *[1 mark]*

(b) The top 10 % of students receive a grade A and the next best 20 % of students receive a grade B. Estimate

(i) the minimum score required to obtain a grade A;

(ii) the minimum score required to obtain a grade B. *[4 marks]*

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3. *[Maximum mark: 6]*

A random variable has a probability density function given by

$$f(x)=\begin{cases} kx(2-x), & 0 \le x \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Show that $k=\frac{3}{4}$. *[4 marks]*

(b) Find $\mathrm{E}(X)$. *[2 marks]*

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4. *[Maximum mark: 6]*

(a) Show that $\frac{3}{x+1}+\frac{2}{x+3}=\frac{5x+11}{x^2+4x+3}$. *[2 marks]*

(b) Hence find the value of k such that $\int_0^2 \frac{5x+11}{x^2+4x+3}\,\mathrm{d}x=\ln k$. *[4 marks]*

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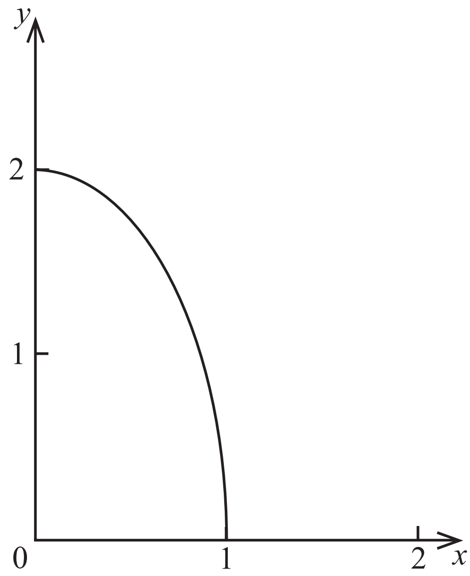
5. *[Maximum mark: 8]*

Consider the part of the curve $4x^2 + y^2 = 4$ shown in the diagram below.



(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y. *[3 marks]*

(b) Find the gradient of the tangent at the point $\left(\frac{2}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$. *[1 mark]*

(c) A bowl is formed by rotating this curve through 2π radians about the x-axis. Calculate the volume of this bowl. *[4 marks]*

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6. *[Maximum mark: 6]*

Let $\boldsymbol{M} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where a and b are non-zero real numbers.

(a) Show that $\boldsymbol{M}$ is non-singular. *[2 marks]*

(b) Calculate $\boldsymbol{M}^2$. *[2 marks]*

(c) Show that $\det(\boldsymbol{M}^2)$ is positive. *[2 marks]*

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7. *[Maximum mark: 8]*

Given that $z_1 = 2$ and $z_2 = 1 + \sqrt{3}\text{i}$ are roots of the cubic equation $z^3 + bz^2 + cz + d = 0$ where $b, c, d \in \mathbb{R}$,

(a) write down the third root, z_3, of the equation; *[1 mark]*

(b) find the values of b, c and d; *[4 marks]*

(c) write z_2 and z_3 in the form $r\text{e}^{\text{i}\theta}$. *[3 marks]*

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8. *[Maximum mark: 8]*

Prove by mathematical induction $\sum_{r=1}^{n} r(r!) = (n+1)! - 1$, $n \in \mathbb{Z}^{+}$.

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9. *[Maximum mark: 8]*

A triangle has sides of length (n^2+n+1), $(2n+1)$ and (n^2-1) where $n>1$.

(a) Explain why the side (n^2+n+1) must be the longest side of the triangle. *[3 marks]*

(b) Show that the largest angle, θ, of the triangle is 120°. *[5 marks]*

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SECTION B

*Answer **all** the questions on the answer sheets provided. Please start each question on a new page.*

10. *[Maximum mark: 22]*

(a) Show that a Cartesian equation of the line, l_1, containing points $\mathrm{A}(1, -1, 2)$ and $\mathrm{B}(3, 0, 3)$ has the form $\frac{x-1}{2} = \frac{y+1}{1} = \frac{z-2}{1}$. *[2 marks]*

(b) An equation of a second line, l_2, has the form $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{1}$. Show that the lines l_1 and l_2 intersect, and find the coordinates of their point of intersection. *[5 marks]*

(c) Given that direction vectors of l_1 and l_2 are $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ respectively, determine $\boldsymbol{d}_1 \times \boldsymbol{d}_2$. *[3 marks]*

(d) Show that a Cartesian equation of the plane, Π, that contains l_1 and l_2 is $-x - y + 3z = 6$. *[3 marks]*

(e) Find a vector equation of the line l_3 which is perpendicular to the plane Π and passes through the point $\mathrm{T}(3, 1, -4)$. *[2 marks]*

(f) (i) Find the point of intersection of the line l_3 and the plane Π.

(ii) Find the coordinates of T', the reflection of the point T in the plane Π.

(iii) Hence find the magnitude of the vector $\overrightarrow{\mathrm{TT}'}$. *[7 marks]*

11. *[Maximum mark: 16]*

A function is defined as $f(x) = k\sqrt{x}$, with $k > 0$ and $x \geq 0$.

(a) Sketch the graph of $y = f(x)$. *[1 mark]*

(b) Show that f is a one-to-one function. *[1 mark]*

(c) Find the inverse function, $f^{-1}(x)$ and state its domain. *[3 marks]*

(d) If the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect at the point $(4, 4)$ find the value of k. *[2 marks]*

(e) Consider the graphs of $y = f(x)$ and $y = f^{-1}(x)$ using the value of k found in part (d).

(i) Find the area enclosed by the two graphs.

(ii) The line $x = c$ cuts the graphs of $y = f(x)$ and $y = f^{-1}(x)$ at the points P and Q respectively. Given that the tangent to $y = f(x)$ at point P is parallel to the tangent to $y = f^{-1}(x)$ at point Q find the value of c. *[9 marks]*

12. *[Maximum mark: 22]*

The complex number z is defined as $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) State de Moivre's theorem. *[1 mark]*

(b) Show that $z^n - \frac{1}{z^n} = 2\mathrm{i}\sin(n\theta)$. *[3 marks]*

(c) Use the binomial theorem to expand $\left(z - \frac{1}{z}\right)^5$ giving your answer in simplified form. *[3 marks]*

(d) Hence show that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$. *[4 marks]*

(e) Check that your result in part (d) is true for $\theta = \frac{\pi}{4}$. *[4 marks]*

(f) Find $\int_0^{\frac{\pi}{2}} \sin^5\theta \, \mathrm{d}\theta$. *[4 marks]*

(g) Hence, with reference to graphs of circular functions, find $\int_0^{\frac{\pi}{2}} \cos^5\theta \, \mathrm{d}\theta$, explaining your reasoning. *[3 marks]*
