22097203

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 1

Thursday 7 May 2009 (afternoon)

2 hours

Candidate session number

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INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 6]*

Consider the complex numbers $z = 1 + 2\mathrm{i}$ and $w = 2 + a\mathrm{i}$, where $a \in \mathbb{R}$.

Find a when

(a) $|w| = 2|z|$; *[3 marks]*

(b) $\mathrm{Re}(zw) = 2\,\mathrm{Im}(zw)$. *[3 marks]*

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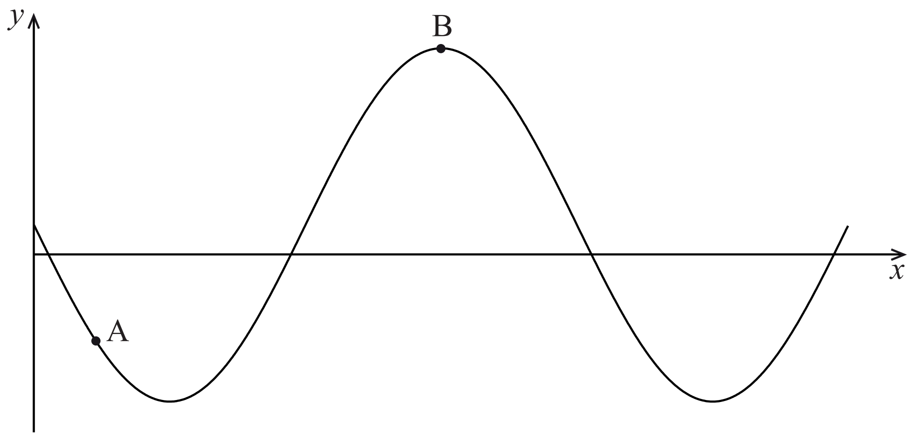
2. *[Maximum mark: 5]*

The diagram below shows a curve with equation $y = 1 + k \sin x$, defined for $0 \le x \le 3\pi$.



The point $A\left(\frac{\pi}{6}, -2\right)$ lies on the curve and $B(a, b)$ is the maximum point.

(a) Show that $k = -6$. *[2 marks]*

(b) Hence, find the values of a and b. *[3 marks]*

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3. *[Maximum mark: 5]*

Let $g(x) = \log_5 \left| 2\log_3 x \right|$. Find the product of the zeros of g.

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4. *[Maximum mark: 6]*

Consider the matrix $\boldsymbol{A} = \begin{pmatrix} e^{x} & e^{-x} \\ 2+e^{x} & 1 \end{pmatrix}$, where $x \in \mathbb{R}$.

Find the value of x for which $\boldsymbol{A}$ is singular.

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5. *[Maximum mark: 5]*

(a) Show that $\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right)=\frac{\pi}{4}$. *[2 marks]*

(b) Hence, or otherwise, find the value of $\arctan(2)+\arctan(3)$. *[3 marks]*

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6. *[Maximum mark: 5]*

The diagram below shows two straight lines intersecting at O and two circles, each with centre O. The outer circle has radius R and the inner circle has radius r.

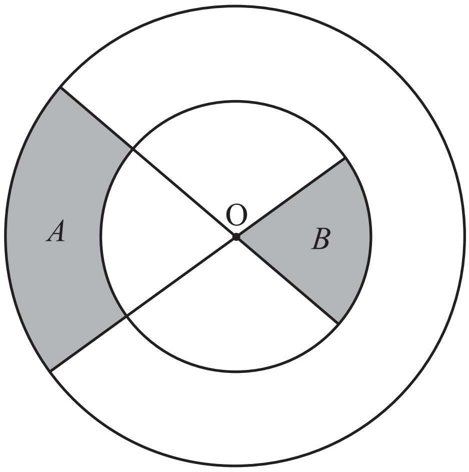

diagram not to scale

Consider the shaded regions with areas A and B. Given that $A:B = 2:1$, find the **exact** value of the ratio $R:r$.

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7. *[Maximum mark: 7]*

Consider the functions f and g defined by $f(x) = 2^{\frac{1}{x}}$ and $g(x) = 4 - 2^{\frac{1}{x}}$, $x \neq 0$.

(a) Find the coordinates of P, the point of intersection of the graphs of f and g. *[3 marks]*

(b) Find the equation of the tangent to the graph of f at the point P. *[4 marks]*

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8. *[Maximum mark: 6]*

A triangle has vertices $A(1, -1, 1)$, $B(1, 1, 0)$ and $C(-1, 1, -1)$.

Show that the area of the triangle is $\sqrt{6}$.

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9. *[Maximum mark: 7]*

(a) Let $a > 0$. Draw the graph of $y = \left| x - \frac{a}{2} \right|$ for $-a \le x \le a$ on the grid below. *[2 marks]*

(b) Find k such that $\int_{-a}^{0} \left| x - \frac{a}{2} \right| \mathrm{d}x = k \int_{0}^{a} \left| x - \frac{a}{2} \right| \mathrm{d}x$. *[5 marks]*

10. *[Maximum mark: 8]*

The diagram below shows a solid with volume V, obtained from a cube with edge $a > 1$ when a smaller cube with edge $\frac{1}{a}$ is removed.

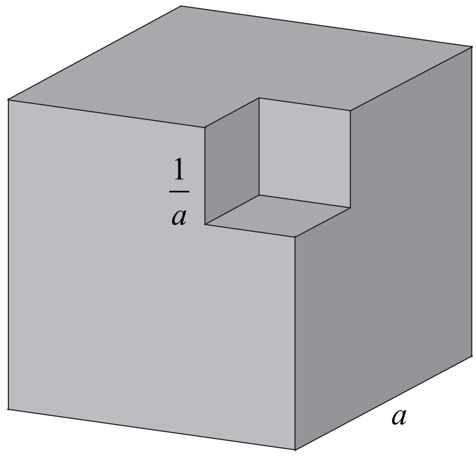

diagram not to scale

Let $x = a - \frac{1}{a}$.

(a) Find V in terms of x. *[4 marks]*

(b) Hence or otherwise, show that the only value of a for which $V = 4x$ is $a = \frac{1+\sqrt{5}}{2}$. *[4 marks]*

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SECTION B

*Answer **all** the questions on the answer sheets provided. Please start each question on a new page.*

11. *[Maximum mark: 20]*

Let f be a function defined by $f(x) = x - \arctan x$, $x \in \mathbb{R}$.

(a) Find $f(1)$ and $f\left(-\sqrt{3}\right)$. *[2 marks]*

(b) Show that $f(-x) = -f(x)$, for $x \in \mathbb{R}$. *[2 marks]*

(c) Show that $x - \frac{\pi}{2} < f(x) < x + \frac{\pi}{2}$, for $x \in \mathbb{R}$. *[2 marks]*

(d) Find expressions for $f'(x)$ and $f''(x)$. Hence describe the behaviour of the graph of f at the origin and justify your answer. *[8 marks]*

(e) Sketch a graph of f, showing clearly the asymptotes. *[3 marks]*

(f) Justify that the inverse of f is defined for all $x \in \mathbb{R}$ and sketch its graph. *[3 marks]*

12. *[Maximum mark: 17]*

(a) Consider the set of numbers $a, 2a, 3a, \ldots, na$, where a and n are positive integers.

(i) Show that the expression for the mean of this set is $\frac{a(n+1)}{2}$.

(ii) Let $a = 4$. Find the minimum value of n for which the sum of these numbers exceeds its mean by more than 100. *[6 marks]*

(b) Consider now the set of numbers $x_1, \ldots, x_m, y_1, \ldots, y_n$ where $x_i = 0$ for $i = 1, \ldots, m$ and $y_i = 1$ for $i = 1, \ldots, n$.

(i) Show that the mean M of this set is given by $\frac{n}{m+n}$ and the standard deviation S by $\frac{\sqrt{mn}}{m+n}$.

(ii) Given that $M = S$, find the value of the median. *[11 marks]*

13. *[Total Mark: 23]*

Part A *[Maximum mark: 9]*

If z is a non-zero complex number, we define $L(z)$ by the equation

$$L(z) = \ln|z| + \mathrm{i}\arg(z),\ 0 \le \arg(z) < 2\pi .$$

(a) Show that when z is a positive real number, $L(z) = \ln z$. *[2 marks]*

(b) Use the equation to calculate

(i) $L(-1)$;

(ii) $L(1-\mathrm{i})$;

(iii) $L(-1+\mathrm{i})$. *[5 marks]*

(c) Hence show that the property $L(z_1 z_2) = L(z_1) + L(z_2)$ does not hold for all values of z_1 and z_2. *[2 marks]*

Part B *[Maximum mark: 14]*

Let f be a function with domain $\mathbb{R}$ that satisfies the conditions,

$$f(x+y) = f(x)f(y),\ \text{for all } x \text{ and } y \text{ and } f(0) \ne 0 .$$

(a) Show that $f(0) = 1$. *[3 marks]*

(b) Prove that $f(x) \ne 0$, for all $x \in \mathbb{R}$. *[3 marks]*

(c) Assuming that $f'(x)$ exists for all $x \in \mathbb{R}$, use the definition of derivative to show that $f(x)$ satisfies the differential equation $f'(x) = k\,f(x)$, where $k = f'(0)$. *[4 marks]*

(d) Solve the differential equation to find an expression for $f(x)$. *[4 marks]*