INTERNATIONAL BACCALAUREATE
BACCALAURÉAT INTERNATIONAL
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N02/540/S(2)M+

MARKSCHEME

November 2002

FURTHER MATHEMATICS

Standard Level

Paper 2

12 pages

Paper 2 Markscheme

Instructions to Examiners

1 Method of marking

(a) All marking must be done using a **red** pen.

(b) Marks should be noted on candidates' scripts as in the markscheme:

- show the breakdown of individual marks using the abbreviations ***(M1)***, ***(A2)*** *etc*.
- write down each part mark total, indicated on the markscheme (for example, ***[3 marks]***) – it is suggested that this be written at the end of each part, and underlined;
- write down and circle the total for each question at the end of the question.

2 Abbreviations

The markscheme may make use of the following abbreviations:

M Marks awarded for **Method**

A Marks awarded for an **Answer** or for **Accuracy**

G Marks awarded for correct solutions, generally obtained from a **Graphic Display Calculator**, irrespective of working shown

C Marks awarded for **Correct** statements

R Marks awarded for clear **Reasoning**

AG **Answer Given** in the question and consequently marks are **not** awarded

3 Follow Through (ft) Marks

Questions in this paper were constructed to enable a candidate to:

- show, step by step, what he or she knows and is able to do;
- use an answer obtained in one part of a question to obtain answers in the later parts of a question.

Thus errors made at any step of the solution can affect all working that follows. Furthermore, errors made early in the solution can affect more steps or parts of the solution than similar errors made later.

To limit the severity of the penalty for errors made at any step of a solution, **follow through (ft)** marks should be awarded. The procedures for awarding these marks require that all examiners:

(i) penalise an error when it **first occurs**;

(ii) **accept the incorrect answer** as the appropriate value or quantity to be used in all subsequent parts of the question;

(iii) award ***M*** marks for a correct method, and ***A*(ft)** marks if the subsequent working contains no further errors.

Follow through procedures may be applied repeatedly throughout the same problem.

The errors made by a candidate may be: arithmetical errors; errors in algebraic manipulation; errors in geometrical representation; use of an incorrect formula; errors in conceptual understanding.

The following illustrates a use of the **follow through** procedure:

Markscheme		Candidate's Script	Marking	
$ 600 × 1.02	***M1***	Amount earned = $ 600 × 1.02	✓	***M1***
= $ 612	***A1***	= $602	×	***A0***
$ (306 × 1.02) + (306 × 1.04)	***M1***	Amount = 301 × 1.02 + 301 × 1.04	✓	***M1***
= $ 630.36	***A1***	= $ 620.06	✓	***A1*(ft)**

Note that the candidate made an arithmetical error at line 2; the candidate used a correct method at lines 3, 4; the candidate's working at lines 3, 4 is correct.

However, if a question is transformed by an error into a **different, much simpler question** then:

(i) **fewer** marks should be awarded at the discretion of the Examiner;

(ii) marks awarded should be followed by '**(d)**' (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(iii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

4 Using the Markscheme

(a) This markscheme presents a particular way in which each question may be worked and how it should be marked. **Alternative methods** have not always been included. Thus, if an answer is wrong then the working must be carefully analysed in order that marks are awarded for a different method in a manner which is consistent with the markscheme.

In this case:

(i) a mark should be awarded followed by '**(d)**' (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(ii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

Alternative solutions are indicated by **OR**. Where these are accompanied by ***G*** marks, they usually signify that the answer is acceptable from a graphic display calculator without showing working. For example:

Mean = 7906/134 ***(M1)***
= 59 ***(A1)***

OR

Mean = 59 ***(G2)***

(b) Unless the question specifies otherwise, accept **equivalent forms**. For example: $\dfrac{\sin\theta}{\cos\theta}$ for $\tan\theta$

These equivalent numerical or algebraic forms may be written in brackets after the required answer.

(c) As this is an international examination, all **alternative forms of notation** should be accepted. For example: 1.7 , 1·7 , 1,7 ; different forms of vector notation such as $\vec{u}$, $\bar{u}$, $\underline{u}$; $\tan^{-1} x$ for arctan x.

5 Accuracy of Answers

There are two types of accuracy errors, incorrect level of accuracy, and rounding errors. Unless the level of accuracy is specified in the question, candidates should be penalized **once only IN THE PAPER** for any accuracy error **(AP)**. This could be an incorrect level of accuracy, or a rounding error. Hence, on the **first** occasion in the paper when a correct answer is given to the wrong degree of accuracy, or rounded incorrectly, maximum marks are **not** awarded, but on **all subsequent occasions** when accuracy errors occur, then maximum marks **are** awarded.

There are also situations (particularly in some of the options) where giving an answer to more than 3 significant figures is acceptable. This will be noted in the markscheme.

(a) **Level of accuracy**

(i) In the case when the accuracy of the answer is **specified in the question** (for example: "find the size of angle A to the nearest degree") the maximum mark is awarded **only if** the correct answer is given to the accuracy required.

(ii) When the accuracy is **not** specified in the question, then the general rule applies:

Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.

(b) **Rounding errors**

Rounding errors should only be penalized at the **final answer** stage. This does **not** apply to intermediate answers, only those asked for as part of a question. Premature rounding which leads to incorrect answers should only be penalized at the answer stage.

Incorrect answers are wrong, and should not be considered under (a) or (b).

Examples

A question leads to the answer 4.6789….

- 4.68 is the correct 3 s.f. answer.
- 4.7, 4.679 are to the wrong level of accuracy, and should be penalised the first time this type of error occurs.
- 4.67 is incorrectly rounded – penalise on the first occurrence.

Note: All these "incorrect" answers may be assumed to come from 4.6789..., even if that value is not seen, but previous correct working is shown. However, 4.60 is wrong, as is 4.5, 4.8, and these should be penalised as being incorrect answers, not as examples of accuracy errors.

Markscheme		Candidate's Script (A)	Marking
(a) $a = 2.31 \times 3.43$ $= 7.9233 = 7.92$ (3 s.f.)	***M1*** ***A1***	(a) $a = 2.31 \times 3.43$ $= 7.9233 = 7.92$	***M1*** ***A1***
(b) $2a = 2 \times 7.9233$ $= 15.8466 = 15.8$ (3 s.f.)	***A1*** ***A1***	(b) $2a = 2 \times 7.29 = 14.58$ $= 14.5$	***A0*** ***A0*****(AP)**
		Total	*2 marks*

Notes: Award ***A1*** for either the exact answer 7.9233 or the 3 s.f. answer 7.92.

In line 3, Candidate A has incorrectly transcribed the answer for part (a), but then performs the calculation correctly, and would normally gain the follow through marks. However, the final answer is incorrectly rounded, and the **AP** applies.

Candidate's Script (B)	Marking		Candidate's Script (C)	Marking
(a) $a = 2.31 \times 3.43 = 7.9233$ $= 7.92$	***M1*** ***A1***		(a) $a = 2.31 \times 3.43 = 7.9233$ $= 7.93$	***M1*** ***A0*****(AP)**
(b) $2a = 2 \times 7.9233$ $= 15.8466 = 15.85$	***A1*** ***A0*****(AP)**		(b) $2a = 2 \times 7.93$ $= 15.86 \quad = 15.8$	***A1*****(ft)** ***A1*****(ft)**
Total	*3 marks*		Total	*3 marks*

Notes: Candidate B has given the answer to part (b) to the wrong level of accuracy, **AP** applies.

Candidate C has incorrectly rounded the answers to both parts (a) and (b), is penalised (**AP**) on the first occurrence (line 2), and awarded follow through marks for part (b).

Candidate's Script (D)	Marking		Candidate's Script (E)	Marking
(a) $a = 2.31 \times 3.43$ $= 7.923 = 7.9$	***M1*** ***A0*****(AP)**		(a) $a = 2.31 \times 3.43 = 7.923$ $= 7.93$	***M1*** ***A0*****(AP)**
(b) $2a = 2 \times 7.923$ $= 19.446 = 19.5$	***A1*****(ft)** ***A0***		(b) $2a = 2 \times 7.93$ $= 15.86$	***A1*****(ft)** ***A1*****(ft)**
Total	*2 marks*		Total	*3 marks*

Notes: Candidate D has given the answer to part (a) to the wrong level of accuracy, and therefore loses 1 mark (**AP**). The answer to part (b) is wrong.

Candidate E has incorrectly rounded the answer to part (a), therefore loses 1 mark (**AP**), is awarded follow through marks for part (b), and does **not** lose a mark for the wrong level of accuracy.

6 Graphic Display Calculators

Many candidates will be obtaining solutions directly from their calculators, often without showing any working. They have been advised that they must use mathematical notation, not calculator commands when explaining what they are doing. Incorrect answers without working will receive no marks. However, if there is written evidence of using a graphic display calculator correctly, method marks may be awarded. Where possible, examples will be provided to guide examiners in awarding these method marks.

1. (i) (a) $\bar{x} = \frac{10}{3}, s_n = 1.83$ ***(G1)(G1)***

[2 marks]

(b) $X \sim \mathrm{P}\left(\frac{10}{3}\right)$ ***(M1)***

For $X = i, f_e = \mathrm{P}(X = i) \times 30$ ***(M1)***

x_i	0	1	2	3	4	5	6	7
f_o	0	5	7	7	2	4	3	2
f_e	1.07	3.57	5.95	6.61	5.51	3.67	2.04	0.97

(A2)

H_0 : "The marks follow a Poisson distribution." ***(C1)***
H_1 : "The marks do not follow a Poisson distribution."

To use the χ^2 test we have to combine classes 0 - 2 and 5 or more, with degrees of freedom equal to 2. ***(A1)***

$$\chi^2 = \frac{(12-10.58)^2}{10.58} + \frac{(7-6.61)^2}{6.61} + \frac{(2-5.51)^2}{5.51} + \frac{(9-7.31)^2}{7.3} = 2.84 < 5.99$$ ***(M1)(A1)***

We don't have enough evidence to reject H_0. Hence we can not reject the claim that the distribution is Poisson with $\lambda = \frac{10}{3}$. ***(R1)***

[9 marks]

(ii) (a) (i) $s = 7.96$ ***(G1)***

(ii) $\bar{x} = \frac{\sum x}{n} \Rightarrow n = \frac{2388}{199} = 12$

$$s = \sqrt{\frac{\sum x^2}{n} - (\bar{x})^2} \Rightarrow s = \sqrt{\frac{45770}{12} - 199^2} = 6.82$$ ***(M1)(A1)***

(iii) $\bar{x} = \frac{12 \times 199 + 10 \times 200.2}{12 + 10} = 199.5$ ***(M1)(A1)***

[5 marks]

(b) (i) $s_p{}^2 = \frac{12 \times 6.82^2 + 10 \times 7.96^2}{12 + 10 - 2} = 59.6$ ***(M1)(A1)***

(ii) H_0: "The samples come from the same population $(\mu_1 = \mu_2)$."
H_1: "The samples don't come from the same population $(\mu_1 \neq \mu_2)$." ***(C1)***

$$t = \frac{200.2 - 199}{7.72 \times \sqrt{\frac{1}{12} + \frac{1}{10}}} = 0.363 < 2.086$$ (number of degrees of freedom is 20) ***(M1)(A1)***

We don't have enough evidence to reject H_0. Hence, we can not reject the claim that the two teams come from the same population. ***(R1)***

[6 marks]

Total [22 marks]

2. (i) $a \circ b \Leftrightarrow 3^a - 3^b = 10k, k \in \mathbb{Z}$

(a) Reflexive *(C1)*

$a \circ a \Leftrightarrow 3^a - 3^b = 10k, k \in \mathbb{Z} \Rightarrow 3^a - 3^a = 10 \times 0, 0 \in \mathbb{Z}$ *(A1)*

Symmetric *(C1)*

$a \circ b \Rightarrow 3^a - 3^b = 10k \Rightarrow 3^{\mathrm{b}} - 3^a = 10(-k) \Rightarrow b \circ a$ *(A1)*

Transitive *(C1)*

$$\left.\begin{array}{l} a \circ b \Rightarrow 3^a - 3^b = 10k \\ b \circ c \Rightarrow 3^b - 3^c = 10l \end{array}\right\} \Rightarrow$$

$3^a - 3^c = 3^a - 3^b + 3^b - 3^c = 10k + 10l = 10(k+l) \Rightarrow a \circ c$ *(M1)(A1)*

[7 marks]

(b) $3^0 = 1 \equiv 1(\mathrm{mod}\,10)$

$3^1 = 3 \equiv 3(\mathrm{mod}\,10)$

$3^2 = 9 \equiv 9(\mathrm{mod}\,10)$ *(M1)*

$3^3 = 27 \equiv 7(\mathrm{mod}\,10)$

So the classes are

$\{0\}, \{1\}, \{2\}, \{3\}$ *(A2)*

[3 marks]

(c) $3^{2002} \equiv 3^2 \equiv 9(\mathrm{mod}\,10)$, so that the last digit is 9. *(M1)(A1)*

[2 marks]

(ii) (a) $a \otimes b = \min\{a, b\} = \min\{b, a\} = b \otimes a$ *(C1)*

[1 mark]

(b) If a and b are negative integers then min $\{a, b\}$ is also a negative integer so the operation is closed. *(C1)*

$(a \otimes b) \otimes c = \min\{a, b\} \otimes c = \min\{a, b, c\} = a \otimes \min\{b, c\} = a \otimes (b \otimes c)$,

so the operation is associative. *(M1)(C1)*

$a \otimes (-1) = \min\{a, -1\} = a$, so the identity element is (–1). *(M1)(C1)*

Every negative integer, apart from (–1), does not have an inverse,

$a \otimes a^{-1} = \min\{a, a^{-1}\} \neq (-1)$. *(M1)(C1)*

[7 marks]

Total [20 marks]

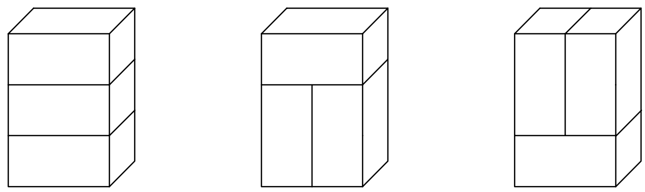
3. (i) Kruskal's algorithm

Edge	Weight	Decision	
FC	1	Add to tree	***(M1)***
CD	2	Add to tree	***(M1)***
DG	2	Add to tree	
BH	2	Add to tree	
HA	3	Add to tree	***(M1)***
DE	3	Add to tree	
AG	4	Add to tree	***(M1)(A1)***

The weight of the minimum spanning tree is 17. ***(A1)***

[6 marks]

(ii) (a)



(C1) ***(C1)*** ***(C1)*** ***[3 marks]***

(b) (i) For a box with dimensions $2\times 1\times (n+2)$, we can start in two different ways. If we put one horizontal brick, there will be a box of dimensions $2\times 1\times (n+1)$ left. ***(R1)(C1)***

If we put two vertical bricks there will be a box of dimensions $2\times 1\times n$ left, so $a_{n+2}=a_{n+1}+a_n$. ***(C1)(AG)***

(ii) The corresponding characteristic equation is

$$r^2-r-1=0\Rightarrow r=\frac{1\pm\sqrt{5}}{2}$$ ***(M1)(A1)***

So, the solution can be written as $a_n=Ar_1^n+Br_2^n$, $A, B\in\mathbb{R}$ ***(M1)***

To find A and B we have to solve the simultaneous equations

$$1=A\left(\frac{1-\sqrt{5}}{2}\right)+B\left(\frac{1+\sqrt{5}}{2}\right), 2=A\left(\frac{1-\sqrt{5}}{2}\right)^2+B\left(\frac{1+\sqrt{5}}{2}\right)^2$$ ***(M1)***

$$\Rightarrow A=-\frac{1-\sqrt{5}}{2\sqrt{5}}, B=\frac{1+\sqrt{5}}{2\sqrt{5}}$$ ***(A1)(A1)***

$$a_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}, n\in\mathbb{Z}^+$$ ***(A1)***

(iii) $a_{15}=987$ ***(A1)***

[11 marks]

Total [20 marks]

4. (a) $\left|\int_1^2 \left(\log_{10} x - \frac{1}{x}\right) \mathrm{d}x\right| = 0.5253816711$ ***(M1)(A1)***

[2 marks]

(b)

x_i	y_i
1	–1
1.25	–0.703089986992
1.5	–0.490575407611
1.75	–0.328390522742
2	–0.198970004336

(A2)

$$A = \frac{1}{12}\left| y_0 + 2(y_1 + y_3) + 4(y_{21} + y_4) + y_5 \right|$$ ***(M1)(A1)***

$$= 0.5255035715$$

$$p = \frac{|0.5255035715 - 0.5253816711|}{0.5253816711} \times 100\ \% = 0.0232\ \%$$ ***(M1)(A1)***

[6 marks]

(c) $\log_{10} x - \frac{1}{x} = 0 \Rightarrow x = 2.506184146$ ***(G2)***

[2 marks]

(d) $x_0 = 2.5,\quad x_1 = 2.512941595,\quad x_2 = 2.498860551$

$x_3 = 2.514192484,\quad x_4 = 2.497511705,\quad x_5 = 2.5156756$ ***(A1)***

We can see that we are getting further away from the zero in every step, so the sequence diverges. ***(R1)***

$$x = g(x) \Rightarrow g(x) = \frac{1}{\log_{10} x} \Rightarrow |g'(2.5)| = 1.097 > 1$$ ***(M1)(A1)***

[4 marks]

(e) $f(x) = \log_{10} x - \frac{1}{x} \Rightarrow f'(x) = \frac{1}{x \ln 10} - \frac{1}{x^2}$ ***(M1)(A1)***

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x_0 = 2.5,\quad x_1 = 2.506172854,\quad x_2 = 2.506184146$ ***(A2)***

[4 marks]

Total [18 marks]

5. (a) If $|\lambda| > 3, (\lambda \neq 0)$ $\dfrac{x^2}{\lambda^2} + \dfrac{y^2}{\lambda^2 - 9} = 1$ then we have an ellipse. ***(A1)(C1)***

If $|\lambda| < 3,\ (\lambda \neq 0)$ $\dfrac{x^2}{\lambda^2} - \dfrac{y^2}{9 - \lambda^2} = 1$ then we have a hyperbola. ***(A1)(C1)***

If $\lambda = 0$ then we have $x = 0$, the y-axis,
while for $\lambda = \pm 3$ we have $y = 0$, the x-axis. ***(C1)***

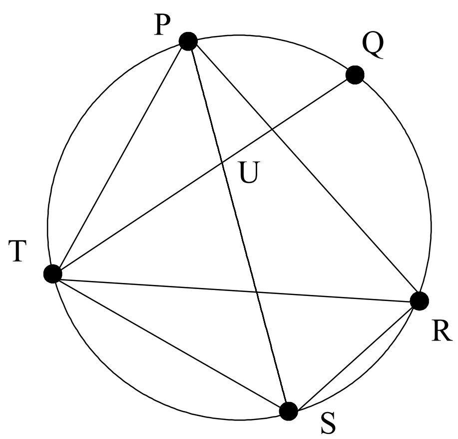
[5 marks]

(b) In the case of an ellipse:
$c^2 = a^2 - b^2 \Rightarrow c^2 = \lambda^2 - (\lambda^2 - 9) = 9$, so $\mathrm{F}_{1,2}(\pm 3, 0)$ ***(M1)(A1)***

In the case of a hyperbola:
$c^2 = a^2 + b^2 \Rightarrow c^2 = \lambda^2 + (9 - \lambda^2) = 9$, so $\mathrm{F}_{1,2}(\pm 3, 0)$ ***(M1)(A1)***

[4 marks]

(ii)



(a) $\mathrm{P\hat{T}U} = \mathrm{R\hat{T}S}$, (arcs of the same length).
$\mathrm{U\hat{P}T} = \mathrm{T\hat{R}S}$, above the common chord [TS]. ***(C1)***
The measures of two angles are equal, therefore the triangles UPT and RTS are similar. ***(R1)(AG)***

[2 marks]

(b) $\mathrm{U\hat{T}S} = \mathrm{P\hat{T}R}$, above the arcs of the same length.
$\mathrm{U\hat{S}T} = \mathrm{P\hat{R}T}$, above the common chord [PT]. ***(C1)***
Two angles are the same, therefore the triangles
UTS and RPT are similar. ***(R1)(AG)***

[2 marks]

(c) $\dfrac{\mathrm{PU}}{\mathrm{PT}} = \dfrac{\mathrm{RS}}{\mathrm{TR}} \Rightarrow \mathrm{PU} = \dfrac{\mathrm{RS}}{\mathrm{TR}} \times \mathrm{PT}$ ***(C1)***

$\dfrac{\mathrm{US}}{\mathrm{TS}} = \dfrac{\mathrm{PR}}{\mathrm{TR}} \Rightarrow \mathrm{US} = \dfrac{\mathrm{PR}}{\mathrm{TR}} \times \mathrm{TS}$ ***(C1)***

$\mathrm{PS} = \mathrm{PU} + \mathrm{US} = \dfrac{\mathrm{RS}}{\mathrm{TR}} \times \mathrm{PT} + \dfrac{\mathrm{PR}}{\mathrm{TR}} \times \mathrm{TS}$ ***(M1)***

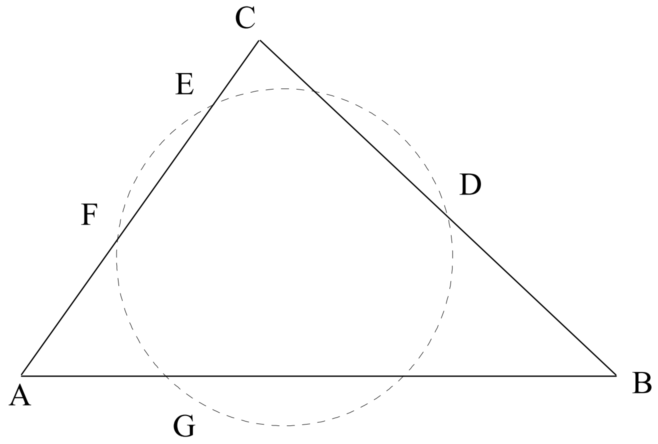
$\Rightarrow \mathrm{PS} \times \mathrm{TR} = \mathrm{RS} \times \mathrm{PT} + \mathrm{PR} \times \mathrm{TS}$ ***(C1)***

[4 marks]

continued…

Question 5(b) continued

(iii)



All the points D, E, F and G are on the nine-point circle, ***(C1)***
therefore DEFG is a cyclic quadrilateral, so we can apply ***(C1)***
Ptolemey's theorem $\mathrm{ED} \times \mathrm{FG} + \mathrm{EF} \times \mathrm{GD} = \mathrm{DF} \times \mathrm{EG}$ ***(R1)(AG)***

[3 marks]

Total [20 marks]