Writing Math Research Papers

A Guide for High School Students and Instructors
Fourth Edition

Robert Gerver

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About the Cover

A treasure map was chosen as the cover theme. Why? The math research journey involves exploration. It involves charting new territory and overcoming obstacles. It involves looking for clues in the mathematics, interpreting those clues, and doing lots of digging to uncover something very rewarding—a new mathematical result!

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For the reasons I've got sunshine on a cloudy day; Linda, Julianne, and Michael

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Introduction

As an avid fan of writing and mathematics, I began giving my students writing assignments as a new teacher in 1977. The initial reaction was "This is *math*— why are we doing English?" As the students completed their math writing projects on varied topics such as automobile insurance, income taxes, stocks and bonds, consumer credit, and more, they began to realize the value of being able to communicate the powerful results mathematics can provide. Students who wrote about surface area, parabolas, the Pythagorean theorem, and other mathematics topics were quick to point out that they internalized the material so much better because in order to explain it they *really* had to understand it. Students concluded that writing is an essential component of any discipline, not just humanities courses. Based on the successes of these early writing experiences, I began to incorporate writing in my mathematics classes on a daily basis. Needless to say, I was very excited when the National Council of Teachers of Mathematics (NCTM), an organization of educators dedicated to the improvement of mathematics instruction, recommended the integration of writing across the mathematics curriculum. Mathematics and writing, to me, have always complemented each other naturally.

Since problem solving is a central focus of math classes, it is clear that writing must be integrated with mathematics courses. Problem solvers have to be able to explain the procedures they used and the solutions they found.

Have you ever been absent from a class and asked a fellow student, "What did we do in math yesterday?" only to hear the response "I know it but I can't explain it"? How did you react? Imagine a detective telling a superior, "I've solved the case but I can't explain my solution"!

Mathematicians must be able to communicate their findings to others so their important results can be used to solve future problems. This is not the only value of writing in mathematics. Writing will also help *you* understand mathematical concepts. As you construct written explanations, you will need to explore and review mathematical concepts in your mind. *Writing Math Research Papers* will help you expand your ability to read about mathematics, explore mathematics, and write about your mathematical experiences. Writing in mathematics also aligns beautifully with the recently implemented Common Core State Standards for Mathematical Practice.

Math Research and the Common Core Standards for Mathematical Practice

The Common Core State Standards were introduced in 2010 and subsequently adopted by over 40 states. These Standards require that these eight principles of mathematical practice (MP1 – MP8) become an integral part of all mathematics curricula:

MP1. Make sense of problems and persevere in solving them.
MP2. Reason abstractly and quantitatively.
MP3. Construct viable arguments and critique the reasoning of others.
MP4. Model with mathematics.
MP5. Use appropriate tools strategically.
MP6. Attend to precision.
MP7. Look for and make use of structure.
MP8. Look for and express regularity in repeated reasoning.

Not all mathematics curricula adequately addressed these requirements. Consequently, thousands of schools all over the United States revised their curricula to incorporate the Common Core recommendations. The Math Research curriculum, as presented in *Writing Math Research Papers*, aligns so perfectly with the eight standards that nothing in the curriculum needed to be "adjusted" to meet MP1- MP8. Math Research is a natural forum for these standards, because it takes an authentic look at how mathematical findings are developed, proven, and extended.

Why Do Math Research?

If you've been successful in mathematics and you enjoy it, you may wonder: "What other mathematical challenges can I explore? How can I tap my enthusiasm in mathematics to further my education?" Math research will provide you with the opportunity to explore mathematics and enjoy the thrill of discovery. Your exploration of a new concept will give you experience in tackling any non-routine problem you may encounter. Throughout this book, you will be introduced to the world of research. By learning problem-solving and research skills, you'll vastly increase your potential to solve all types of problems.

You already possess many of the tools necessary to do exploratory mathematics. Perhaps you have done some problem solving in your math class. Your math research paper will be a major project—you will not finish it in one day or even one week. It will require you to read, write, think, and investigate mathematical ideas. It will improve your writing, reading, and oral communication in other subjects as well. It will empower you to create, conjecture, challenge, and question according to your ability, motivation, priorities, and schedule. Research will expose you to the beauty and practicality of mathematics and reward you with the tremendous feeling of uncovering a result previously unknown to you. Communicating your experiences through the writing of the research paper will help you understand and appreciate the mathematics you've explored as well as help others learn about your findings.

Your instructor can serve as coach, and this book can serve as a guide, but the driving force behind your research is your own motivation. You will become an expert on your topic because you will spend a great deal of time exploring it. You will learn mathematics by *doing* mathematics.

What's in This Book?

Often, when mathematics "term papers" are assigned, students are given a list of topics and a due date. Specific instruction on reading, extending, and writing mathematics may not be offered. Writing mathematics and doing mathematics research involve sophisticated skills that are not innate. The purpose of *Writing Math Research Papers* is to introduce you to these skills and give you logical, sequential direction in developing valuable skills that will last a lifetime. We examine the purpose of each chapter here to better acquaint you with how these skills will be developed.

Chapter 1: Mathematics: Shouting Questions and Whispering Answers

As the title implies, as you engage in mathematical thinking it is natural for many questions to arise. Many of these questions are open-ended, challenging inquisitions. No matter how deeply you delve into a topic, new questions will always surface; hence, mathematics *shouts* questions. Finding the answers to these questions requires time, effort, and skill. The answers do not surface as quickly as the questions; hence, mathematics *whispers* answers. The chapter contains some examples that show that even basic arithmetic concepts can inspire intriguing questions.

Chapter 2: Finding a Topic

Many students are first exposed to research in history and science courses. Some students have even written research papers and conducted research experiments in these fields. The difference between a report and a research paper is discussed in this chapter. Often, students who write math papers pick a topic that is too broad. As this chapter points out, topics for research papers must be specific. There are many sources for succinct topics for research papers. Mathematics journals written for instructors and math enthusiasts feature short articles that describe particular mathematical concepts. These articles, usually three to six pages long, can provide an excellent springboard for your paper. Chapter 12 lists journals you can use as sources for your paper's topic. Other suggestions for finding topics are also discussed in this chapter. In summary, your research paper will build upon a short article, problem, or idea, rather than attempt to condense information from several immense library books.

Chapter 3: Problem Solving: A Prerequisite for Research

Problem solving has always played an integral part in mathematics courses. You will encounter non routine problems on the job and in everyday life, and the solutions to these problems will be crucial. Solid problem-solving experience is advantageous. This chapter reviews problem-solving strategies you may have learned in some of your previous math courses. Familiarity with these strategies will help you answer some of the questions that arise as you carry out your research. Additionally, you will use these problem solving strategies throughout your entire life!

Chapter 4: The Importance of Clear Communication

This chapter underscores the need to express your thoughts coherently and comprehensively. Some of the mathematics you are researching may be pretty difficult to understand, so it needs to be explained clearly to the reader. Communication is an art and a skill just like playing the piano, pitching, or plumbing, and it can be learned, practiced, critiqued, improved, and mastered.

Chapter 5: The Notes You Take in Math Class

This chapter provides several activities for writing about mathematics that you can use to improve your writing even if you don’t intend to write a math research paper right away. They will improve your note-taking in class as well as help you communicate your thoughts clearly whenever you write about mathematics. These writing activities concentrate on mathematics you are already familiar with from your core math class, so you can concentrate solely on improving your writing skills.

Chapter 6: Technical Writing Techniques

This chapter gives specific writing strategies that can be used to achieve the characteristic formatting you saw throughout the projects in Chapter 4. Technical writing has its own set of rules, which makes it slightly different from other forms of writing you have learned in school, including creative writing, poetry, document-based essays, etc.

Chapter 7: Conjectures, Theorems, and Proofs

The similarities and differences between history, science, and mathematics research and the role of proofs in each are discussed in this chapter. In your mathematics reading, you will encounter proofs. You should try to read through them and understand them so you can explain them in your paper. You may come up with some questions of your own and even make conjectures (hypotheses) about the answers to these questions. As you become more experienced, you may actually prove your conjectures with original proofs. This chapter will expose you to the role that proofs can play in your research. Your experience with proof will be invaluable to you in future math courses.

Chapter 8: Reading and Keeping a Research Journal

You might not understand how to turn a short article into a research paper, but as you read through this chapter you will see that you might be able to read fifty pages of a novel in less time than it takes to carefully read and digest one page or even one paragraph of mathematics. Specific tips for reading mathematics journal articles are given. Because you will not start writing the formal paper until part of your research is completed, you will need a comprehensive set of notes on your research. Thus, you will keep a journal as you read. You will keep anything your write about, including scrap notes and errant trials.

Chapter 9: Components of Your Research Paper

Chapters 3, 4 and 5 introduce you to writing mathematics, and Chapters 6, 7, and 8 instruct you in how to conduct your research in a logical fashion. Chapter 9 helps you pull it all together for the formal paper. The parts of the research paper are discussed. As you read this chapter, you will also refer to Chapter 11, which features samples of actual student work that will help you get a feel for how the formal paper should appear.

Chapter 10: Oral Presentations

Many students are given a chance to present their research at special events and mathematics competitions. Giving an oral presentation requires extensive planning, visual aids, and practice. This chapter will give you tips in staging a polished, professional-quality presentation that reflects the high quality of your research.

Chapter 11: Sample Pages from Student Research Papers

Chapter 11 shows some actual pages from math research papers done by high school students. These samples will help you get used to the formatting, language style, tables, diagrams and other idiosyncrasies inherent in technical writing. You can use them as references when you write.

Chapter 12: Some Suggested Resources

This chapter offers suggestions for specific journals, books, periodicals, and websites that can be used to find topics, enhance papers, extend papers, and run your research course. Recommendations and citations for specific journal articles appropriate for high school students are also included.

Chapter 13: A Guide for Instructors and Administrators

Creating and implementing a new program such as a math research course is a major undertaking. Preparing scheduling, logistics, enrollment, technological necessities and physical space require careful planning. Chapter 13 includes suggestions for instituting your math research program.

After reading the purpose of each chapter, you may wonder: "Will doing a math research paper strengthen my mathematics education? If so, how?"

Improving Your Mathematics Education in Many Facets

The skills you will acquire during your research project are considered very valuable by mathematics educators and can be applied to other mathematics courses as well as to other disciplines. Your research project reflects an effective, practical incorporation of many mathematics education principles in a format that will allow you to experience some of the power of the various teaching and learning strategies used in mathematics. Your research topic may be a mathematical application to another discipline or a pure mathematics topic—one that advances knowledge about a certain mathematics concept rather than solving a practical problem in another discipline. In either case, the research strategies you learn will be valuable in virtually all mathematical situations you encounter. Chapter 7: Finding a Topic will help you decide what road you should take. You will be traveling down a mathematical highway illuminated by communication, reasoning, connections, problem solving, representation and modeling.

Communication and Your Research

Researchers need to be communications-minded. They must read, attend lectures, listen, write clearly, and make oral presentations of their work. Researchers need to digest information, process it, perform their work, and report their findings. You will benefit from others' ability to communicate because, early in your research, you will be reading about your topic. Chapter 9: Reading and Keeping a Research Journal examines specific reading skills you can adopt to help in reading technical writing. As you read, you will ask yourself informal questions to help you understand passages as well as formal questions as mathematical extensions of your readings. As you explain to others where your questions lead you, your explanations must be clear to outside readers. Good communication skills are essential to effective presentations, both written and oral. Chapters 3, 4, 5, 9, and 10 delve into the communication arts with respect to your research. Above all, since clarity is paramount. Your organization, definitions, questions, proofs, conjectures, and explanations must be well written and logically organized. You'll need to reflect on each stage of your work carefully before you can put it into your own words. You'll need to use notation, graphs, and other forms of representation effectively in order to convey mathematics succinctly. Becoming adept at communicating your work will allow others to benefit from the knowledge you've acquired. Other student-researchers may want to continue your work, extend it, alter it, or make new investigations. They will rely on your communication skills.

Reasoning and Your Research

Students need to develop an ability to make conjectures, interpret claims, justify claims, and communicate their findings. As you read through your research articles, you will need to follow the arguments presented by the authors. You must make sure that their arguments are valid and that you understand the logic used by testing their claims and working through their proofs step-by-step. You will formulate ideas based on patterns, your mathematical intuition, and the mathematical tools you have acquired in your coursework. Making conjectures requires reasoning—conjectures aren't guesses but rather hypotheses that, whether true or false, can reasonably be tested. Mathematicians make many conjectures that turn out to be false. If, after working with a conjecture, you suspect that it is false, you might try to find a counterexample—a single case in which the conjecture is not true—or explain theoretically why your original suspicions were not true in every case. If you are convinced that your conjecture is true, you may try to construct a proof—a valid argument that your hypothesis is indeed a theorem. There are different types of proofs for different conjectures. You might read a statement that is not proved and decide to construct a proof on your own. This proof then becomes part of your research. Chapter 6 gives an overview of the use of proofs in mathematics.

Connections and Your Research

As you explore your paper's topic, you will have the chance to integrate different branches of mathematics in your research. Students doing research on a geometry topic might use algebra, logic, calculus, trigonometry, set theory, and more to create proofs and give explanations about their topic. Tapping the different fields as they are needed requires knowledge and discretion, as well as a well-equipped mathematical tool kit. In this respect, your paper is different from the study of a single unit in a math course. You may need to learn part of a topic on your own (with the assistance of your instructor and an appropriate textbook) because it can help your research. You may be able to test or prove one of your conjectures in two different ways—for example, using coordinate geometry and plane geometry. As part of your research, you might discuss which method was easier, better, faster, shorter, more intuitive, and so on.

If your paper deals with a mathematics application to another discipline such as psychology, business, or science, you will be making connections not only within mathematics but between mathematics and the discipline you're modeling. When researching an applied-math topic, you need to become knowledgeable about the discipline you are researching as well as the mathematics you are using. The connections are seemingly endless:

- What must an architect know about an ellipse in order to design a whispering gallery?
- What is the shape of the suspended cables of a suspension bridge?
- How are seismographs, rates, and circles used to find the epicenter of an earthquake?
- How can mathematics be used to find the area of an irregular shape such as a golf green?
- How are graphs and statistics used to predict economic trends?
- How can doctors use conic sections to break up kidney stones without invasive surgery?
- How are paraboloids used to create telescopes that can take pictures deep into space?

As society becomes more technologically oriented, mathematics assumes a more prevalent role in the progress of other disciplines. A connection to mathematics is an essential component of the research that will advance knowledge in other fields. Use your communication skills to help your readers make connections *in* your work and *to* your work.

Problem Solving and Your Research

Your readings will include passages that you don't immediately understand. As you reread certain sentences several times, you will need to employ your problem-solving skills to figure out their meaning. The passage you are having trouble with in effect becomes its own problem. Attack it with determination. Your readings will include many such hurdles. You might even create some hurdles yourself, since each concept you understand and internalize may breed more questions and possible extensions. Such questions and extensions are really new problems. Pose them to your readers to investigate on their own, or raise their solutions and address their solutions in your research. Chapter 2 provides you with an overview of problem-solving strategies.

Representation, Modeling, and Your Research

You are already familiar with mathematical representation. If you saw a foreign mathematics textbook, you'd realize how adept you are at reading mathematical thoughts. Each year, I expose my students to a few pages of mathematics textbooks written in other languages, even languages that use other types of letters. The students are always amazed at how much they can understand in a French, Hebrew, or Chinese textbook. How is this possible?

Mathematics is full of representations. All the symbols and diagrams we use represent something, and these representations are often universal. A representation could have some interesting historical origin, or it could simply be a convenient shorthand used to represent some fact. (Think of how the percent symbol, %, actually incorporates the 1 and two 0s from the number 100.) A model is a representation. Some models are diagrams—physical representations—for instance, a perspective drawing of a cube. Other models use symbols. The equation $C = 3v + 10$ could be used to model the cost C of downloading v videos from a movie club each year, if the movie club has a $10 annual membership fee, and they charge $3 for each download. In statistics, you can represent an average as a mean or a median, and you must interpret the situation to apply the best representation. You can oversimplify a problem or actually do it incorrectly by using an incorrect representation. Representations are the tools that expand your capacity to think mathematically. They are the numbers and the concepts behind them; they are algebraic symbols, expressions, equations, and graphs; they are geometric terms and models; they are statistical formulas and displays. Representation is essential to understanding and communicating mathematics and to applying mathematics to real-life situations. To describe a situation mathematically, you need to represent the situation using the language of mathematics.

Your research in mathematics may include investigating notation new to you. You may also invent new notation to help you organize your findings as you answer a research question. You will use representations to communicate your research results. As you research and present your findings, you will be choosing among different forms of representation. You can use graphs, pictures, diagrams, lists, and narratives. With new technology, including graphing calculators and dynamic geometry software, you might find that your investigation leads you to new notations and terminologies. Think of all the words in your daily technical vocabulary that most people didn't know twenty years ago—hashtag, Twitter, smart phone, download, instagram, tweet, and so on. Think of what "closing a window" meant thirty years ago, and how the same phrase today connotes working on a computer! New notations, processes, and terms will lead you to form new questions.

Chapters 3–6 will require you to examine the use of mathematical representation as you engage in your research, and the culmination of your research—your oral presentation—will depend on good representation to be clear and accurate.

How to Use *Writing Math Research Papers*

Ideally, you will use *Writing Math Research Papers* in a group setting, with your classmates and instructor giving you feedback and coaching. If you are planning to write a math research paper, you and your classmates should read *Writing Math Research Papers* gradually as you go through the research process. The following time line is offered as a general guide to your project. Let's look at how a nine-month school year could be scheduled.

Month	Activity
1, 2	Read and discuss *Writing Math Research Papers,* Chapters 1–7, and 12. Find a topic and an article. Make copies of your article, and start a Math Author Project using a topic from your core mathematics class. Start your bibliography. These activities do not have to be finished within the first two months—they need to be started.
2, 3, 4, 5	Begin reading your article, taking notes, testing claims, and looking for patterns. Keep a journal of all your writings. Save your annotations of the article. Begin periodic (usually weekly) consultations. Read Chapter 9.
4, 5, 6, 7	Continue reading your article, writing up findings, keeping your journal, and consultations. Write your Problem Statement. Compile all of your findings to date for your Related Research section.
7, 8, 9	Read Chapter 10 and start planning your oral presentation. Present paper to an audience and record the presentation. Watch it and critique it.

As you progress through each stage of your research, you should reread previous chapters and use them as a reference. It is based on a two-semester course, but can be adjusted to meet your specific situation. The time line can help you prorate your time if you are not writing your paper over a full two-semester period. The amount of time you have, the amount of coaching you receive, and the topic you choose will all affect the amount of time you spend on any one activity. Always remember that research projects require flexibility. You can never "guarantee" when you are going to come up with certain findings or proofs. However, you *can* guarantee the time, trials, and effort you put in. Following are some suggestions for other ways you could use the book:

- If you are not sure whether you are going to write a research paper, reading through Chapters 1, 2, 8, and 9 can help you decide if you should undertake such a project.
- If you want to improve your mathematics writing skills by practicing writing about the mathematics you have already learned, read about the writing mathematics activities in Chapters 4, 5 and 6.
- If you would like to improve your ability to take notes in mathematics classes, read and do the activities in Chapters 4, 5 and 6.
- If you are an instructor or administrator planning a course in problem solving and/or mathematics research, read the entire book, with an emphasis on Chapter 13: A Guide for Instructors and Administrators.
- If you are an instructor coaching a student who is doing an independent- study

math research paper, Chapters 2 and 12 will help you and your student find a topic. Chapter 13 will help you understand the role you can play as this student's mentor.

- If you are a pre-service teacher in a college mathematics education program, you should read the entire book to familiarize yourself with what your future students may be doing. Perhaps you never wrote a math research paper, and most of the process is new to you, too. You may want to try writing your own research paper following the steps outlined in the book. Keep in mind that Chapters 4, 5 and 6 have excellent tips you can employ to help all of your students take better notes. Chapters 12 and 13 will help you with resources and classroom logistics. As you enter the world of mathematics writing and mathematics research, keep in mind that fellow educators and students are always interested in your ideas, successes, and suggestions.

"What we have to learn to do, we learn by doing."

--Aristotle

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Chapter 1

Mathematics: Shouting Questions and Whispering Answers

Every day, dozens of questions pop up in your head. Dozens. Literally. You may not even realize it. Some are answered immediately, while some are never answered. Others require you to search for an answer.

- What's for lunch today?
- Do we have Spanish homework?
- What time is it?
- Was Dylan at track practice?
- How much battery life does my phone have?
- Where did I park my car?

The list is seemingly endless. Some of the answers are ones you might have expected, while others surprise you.

Many times in your mathematics classes you have asked questions. These are traditionally content-related questions. When you raise your hand in class to ask a question, you always expect to "get the answer" from your teacher immediately. You would be stunned if your teacher ever replied "I don't know." When you have trouble with your homework, you can always count on "getting the answer" the next day in class.

Unfortunately, the speed at which we "get answers" is really not an accurate microcosm of how mathematics develops. We are all so used to instantaneous answers, with our microwave ovens, smart phones, tablets and iPods that we have too little experience having to wait days, weeks, months, years or even centuries to get answers to questions that arise. Of course it is convenient to have your pizza ready in 45 seconds, or send an e-mail across the world in 10 seconds. But one of the prices we pay for having all this instant gratification is that we have too little practice in exercising patience and waiting to get an answer.

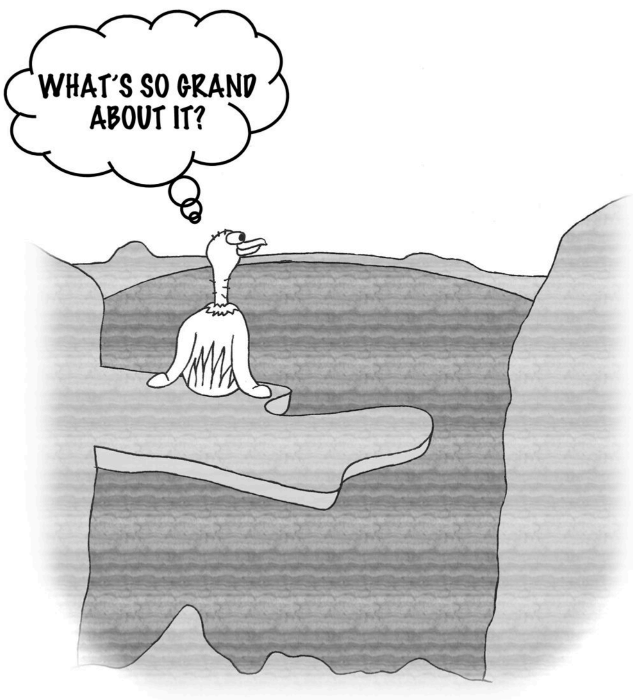
The high school mathematics curriculum is designed to expose you to many facets of mathematics, and give you some basic tools for your future. In that sense, it is quite effective. Yet, school mathematics is somewhat "reputationally challenged." Students often don't see the "need" for the law of cosines, imaginary numbers, or logarithms. What is probably the most common question asked in high school mathematics classes?

When am I Ever Going to Need This?

Did you ever ask, or hear a fellow student ask, in a mathematics class, "When are we ever going to *need* this?" Why does mathematics always seem to invite this cynicism? Think about your other high school subjects. Do you ever *use* Shakespeare? Medieval history? Fur Elise? Rembrandt? Probably no more or less so than you use mathematics. Does a subject have to "fix something" to be considered worthwhile?

"It is hard to believe that someone flying over the Grand Canyon for the first time could remark, "What good is it?" Like the Grand Canyon, mathematics has its own kind of beauty and appeal for those who are willing to look."

—Harold R. Jacobs, *Mathematics: A Human Endeavor,* 1994



Virtually everyone uses geometry, arithmetic, measurement, and statistics as citizens, taxpayers, consumers, and homeowners. Realistically, not everybody uses trigonometry, quadratic equations, or linear regression in their daily lives, or even in their careers.

However, if you appreciate e-mail, your cell phone, GPS, tablet, MRIs, computers, air travel, automobiles, and other modern conveniences, be thankful that people are exposed to high-level mathematics and that some become math enthusiasts, continue their study, and go on to develop essential pieces of the puzzles that eventually become the modern conveniences you probably feel you couldn't live without. The desire to satisfy a curiosity has led to the greatest inventions of our time—all because inspired people asked questions and were motivated enough to seek the answers.

Mathematics is an extremely useful science, with applications in health, economics, astronomy, physics, sports, medicine, and many other fields. However, like art and music, mathematics is beautiful in its own right—pure mathematics can be just as inspiring as the Mona Lisa. Finding new patterns and making new discoveries is very exciting. Exciting?! Authentic mathematics actually has much in common with rock climbing, rock concerts, and rocket launches. How are these similar?

- They engage millions of people.
- They are challenging for the participants.
- They can be exciting.
- Each has a degree of uncertainty. There are many unknowns in each scenario.
- Each offers an opportunity for teamwork.
- There is exhilaration and satisfaction when a given task in each element is completed successfully.

Perhaps you think of algebra when you hear the term "unknowns." Often in algebra, we are trying to find the unknown value of a variable. In our current context, we are using the term "unknowns" to talk about problems, facts, and theories that are currently unsolved or not yet discovered. Why are we talking about unknowns in mathematics? Hasn't all mathematics already been discovered? The answer to this question is a resounding no! There is much yet to be uncovered in all branches of mathematics. Note that the experience of exploring the unknown does not have to deal with unsolved problems. A problem that is new to *you* is an unsolved problem for you. Much beautiful, fascinating, surprising, and useful mathematics is new to you and waiting to be explored.

What Mathematics Is Unknown to You?

Most likely, your research, especially at the beginning stages, will be confined to mathematics that is known in the field but not known to you specifically. Mathematics journal articles are a terrific source for such mathematics topics. (You will learn more about these article in later chapters throughout *Writing Math Research Papers*). As you read about and learn mathematics that is not covered in the typical high school curriculum, you will formulate questions. You can attempt to find the answers to these questions as you extend your research beyond what the article initially taught you!

Let's examine a problem about consecutive integers and their sums from a journal article (Olson, 1991). Notice that the following numbers can be expressed as sums of consecutive integers:

$21 = 6 + 7 + 8$

$7 = 3 + 4$

$30 = 4 + 5 + 6 + 7 + 8$

$2 = (-1) + 0 + 1 + 2$

$5 = (-4) + (-3) + (-2) + (-1) + 0 + 1 + 2 + 3 + 4 + 5$

Can *any* whole number be expressed as the sum of consecutive integers? Can any whole number be expressed as the sum of consecutive counting numbers? What do you think? Try expressing the counting numbers from 1 through 10 as sums of consecutive integers. Can you express any of these numbers as consecutive-integer sums in more than one way? Although the answers to these questions are "known," if the answers are unknown to you, you can investigate these problems as unsolved problems in mathematics. You may actually come up with a question that has *never been asked! Ever!* You will truly be "doing" mathematics as you try to uncover the solutions. Let's continue by investigating properties of a very famous sequence.

Perhaps you are familiar with the Fibonacci sequence. The Fibonacci sequence is a sequence of positive integers that begins with 1, 1. Subsequent terms are found by adding the two previous terms. The first fifteen terms of the Fibonacci sequence are listed here:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610 . . .

What patterns do you notice? Can you create questions about the sequence? Let's look at some patterns and questions to become accustomed to viewing mathematics as an "open" rather than a "closed" science.

Some questions:

- Are there infinitely many Fibonacci numbers?
- Are there more odd than even Fibonacci numbers?
- Are any Fibonacci numbers perfect squares?
- Are any Fibonacci numbers prime?

Some patterns:

- The numbers are increasing.
- Every fifth Fibonacci number is divisible by 5.
- The difference between consecutive Fibonacci numbers increases as the numbers increase.
- The sequence seems to have two odd integers followed by one even integer.
- If two consecutive Fibonacci numbers are squared, the sum of these two squares is also a Fibonacci number.
- The ratio of one term to the previous term is always less than or equal to 2.

Some surprises:

- There are as many Fibonacci numbers are there are positive integers, yet the Fibonacci numbers are a subset of the positive integers!

- Pick three consecutive Fibonacci numbers. Square the middle number. From that square, subtract the product of the first and third numbers you picked. The answer is either 1 or -1.

- Let Fi represent the i^{th} Fibonacci number. Create a sequence of Fibonacci numbers of the form

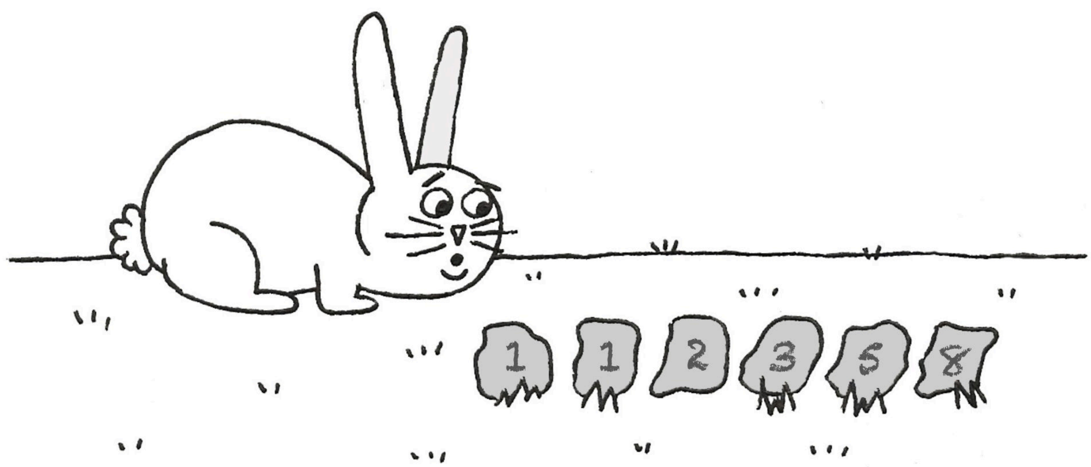
$$\frac{F_i}{F_{i+1}}.$$

  As *i* increases, the fractions of this form approach an irrational number 0.382…

- To find the 796^{th} Fibonacci number, you could add the 795^{th} Fibonacci number to the 794^{th} Fibonacci number. But imagine how long this would take! Let F_n represent the n^{th} Fibonacci number. There is a formula that can generate the n^{th} Fibonacci number directly, without needing *any* of the previous Fibonacci numbers:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

This formula would certainly not be obvious by just looking at the sequence, like some of the other patterns were. How many of the patterns did you notice? What other patterns did you notice? How could conjectures based on these patterns be proved? Do you think there are other patterns you haven't noticed yet? The answers to questions, the formulation of conjectures, and the proofs of those conjectures may not be uncharted territory in mathematics, but they may be unsolved problems for you. Solving problems that are original for you allows you to experience the discovery process. The freshness of new, often fascinating ideas can team up with the mathematics you have already learned in school to provide you with a challenging, rewarding trip into the world of real mathematics!



Becoming Inquisitive

As you ponder virtually any mathematics topic, questions will naturally arise. Even seemingly simple ideas contain unlocked secrets. The purpose of this section is to show you that questions exist everywhere and that you should always formulate questions about the mathematics you are doing. The main issue here is the questions; in fact, some of the answers are deliberately omitted. You might try to answer them as part of a research project or simply as a problem-solving exercise. Concentrate on the *questions* in each example. Try to think of your own questions, too. The first example involves elementary subtraction.

Take a look at basic subtraction of three-digit integers by examining the following subtraction example:

$$\begin{array}{r} 954 \\ \underline{-459} \\ 495 \end{array}$$

Did you notice that the minuend, the subtrahend, and the difference were all formed from the same three digits, 4, 5, and 9? For what other three-digit numbers does this happen? Did you notice that the minuend's digits were in descending order and the subtrahend's in ascending order? Let's try another example of this type.

$$\begin{array}{r} 753 \\ \underline{-357} \\ 396 \end{array}$$

Notice that the answer does not use the same three digits, but the difference's outside numbers do add up to the inside number. Do the outermost digits of the difference always add up to the middle digit? How many three-digit numbers are there? Must all of them be tested individually to answer these questions? Did you ever think that a simple subtraction example could launch so many questions? Try other examples of this type and see what kind of patterns you can find. Can you think of other questions based on the examples you generate?

When you were in elementary school, you learned how to multiply fractions:

$$\frac{1}{2} \times \frac{3}{5} = \frac{3}{10}$$

The procedure was so predictable that many students were able to guess it without prodding from the teacher. After learning multiplication, you learned how to divide fractions:

$$\frac{25}{32} \div \frac{5}{4} = ?$$

Based on the multiplication algorithm, your first guess might be to divide the numerators and then divide the denominators. Is this correct?

$$\frac{25}{32} \div \frac{5}{4} = \frac{5}{8}$$

The traditional procedure requires you to find the reciprocal of the divisor and multiply:

$$\frac{\overset{5}{\cancel{25}}}{\underset{8}{\cancel{32}}}\times\frac{\overset{1}{\cancel{4}}}{\underset{1}{\cancel{5}}}=\frac{5}{8}$$

Notice that the answers match. Is this a coincidence, or is the first procedure valid? Does it always work? The first procedure was correct! Then why is the more unnatural second procedure a staple in elementary schools? Does the second procedure have advantages? In fact, the first procedure always works, but it can become cumbersome when the numerator and denominator of the divisor are not factors of the numerator and denominator of the dividend, respectively, as shown in the following example:

$$\frac{26}{31}\div\frac{5}{4}=?$$

This examples shows the logical reason for the universality of the second "keep, change, flip" procedure; however, there is plenty of room for conjectures, trials, errors, questions, and interesting discussion. It is never too early to develop a feel for questioning and probing. Today's mathematics courses are encouraging this.

The procedure for addition of two fractions involves common denominators. Most elementary-school students' best guesses of the sum of two fractions probably resembles

$$\frac{9}{10}+\frac{17}{20}=\frac{26}{30}$$

We know this is not the correct answer. Why not? Would this method ever work? For what fractions? Can you think of a scenario in which the above arithmetic algorithm might make sense?

Julianne received 9 out of 10 on yesterday's French quiz. She received 17 out of 20 on today's French quiz. What fraction can be used to convert her cumulative quiz scores to a percent for the two quizzes?

The answer is 26/30. On the number line, we know that $9/10 + 17/20 \neq 26/30$. However, the intuitive but incorrect attempt to solve this addition problem does have an application. Perhaps a symbol other than "+" could be created to indicate when the above algorithm can appropriately be applied to two fractions. For example:

$$\frac{9}{10} \# \frac{17}{20} = \frac{26}{30}$$

In your math research, you will be free to make up new terms and symbols, as long as they are well-defined. Notice this new symbol has the "flavor" of addition but is not the traditional addition symbol.

Learning that something *isn't* true is a valuable part of probing in mathematics. It creates a natural motivation to search for a correct procedure. If after much trial, error, and questioning you discover a correct procedure on your own, you experience *real* mathematics. Let's examine elementary fractions further.

Remember when you learned how to "reduce" fractions? The word is in quotes because we don't make a fraction smaller when we reduce it but rather make the numerator and denominator smaller. This procedure is often referred to as "simplifying" the fraction. We say that 11/33 is equal to 1/3 in simplest form. Even this term is sometimes misleading. You could argue that, although 19/25 is the "simplified" version of 76/100, in fact 76/100 is "simpler" to understand because we are so used to comparing numbers to 100 (converting to per cents). If you answered 19 out of 25 questions correctly on a quiz, the first thing you would probably do is convert it to 76% to get a better feel for how you did. Would you now question which fraction is actually simpler? Did you question this in elementary school?

Look at the following procedure for simplifying fractions:

$$\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$$

$$\frac{2\cancel{6}}{\cancel{6}5} = \frac{2}{5}$$

$$\frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5}$$

$$\frac{4\cancel{9}}{\cancel{9}8} = \frac{4}{8}$$

Are the answers correct? Yes. Does this procedure always work? When does it? How would you attempt to answer these questions?

Let's examine a problem involving decimals and per cents.

Courtney bought a lawn mower that was discounted 25% because it was a damaged floor model. The store was also having a 10%-off sale. The cashier first took 10% off the original price and then took 25% off the reduced price. Courtney requested that the store first take 25% off the original price and then take 10% off the reduced price, since she felt that 25% of the larger price would bring a greater discount. Does it matter which discount is taken first? Is Courtney's discounted price equivalent to a 35% discount? How would you attempt to answer these questions?

Let's examine another problem involving decimals.

Jordan walks into a convenience store and purchases four items, with four different prices. The cashier multiplies the four prices correctly, and says, "That'll be $7.70."

Jordan replies, "You were supposed to *add* the prices, not multiply them!"

The cashier adds the four prices correctly, and says, "You are right. That'll be $7.70."

What were the prices of the four items?

Try writing a program on your graphing calculator to solve this. Could you do it using trial and error? (Answers: $1, $1.25, $1.60, $3.85)

Doing math research requires you to become adept at asking questions. Math research is a series of questions, conjectures, and answers based on the investigation of a topic. Conjectures are building blocks of good problem solving. Did you realize that the mathematical discoveries with which you are familiar took many attempts to discover? Can you imagine the elation and satisfaction people feel when they encounter a problem, create key questions, answer the questions, and solve the problem? A great deal of challenge and excitement awaits you as you enter your mathematical unknown via your research project.

Unknowns in Mathematics

Although your research may not focus initially on solving unknown problems in mathematics, you might be surprised and excited to find out that many unsolved problems can be explained on a very elementary level. There are many unsolved problems in mathematics. Perhaps you think that unsolved problems exist only in the high levels of advanced college mathematics. That is a misconception. All areas of mathematics have unsolved problems. We will investigate three unsolved problems related to prime numbers.

Prime numbers are numbers that have exactly two divisors—themselves and the number 1:

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, \ldots\}.$$

Are there infinitely many prime numbers? How would you go about answering this question? Would your answer be influenced by a person giving you a computer-generated list of the first 25,000 prime numbers?

There are, in fact, infinitely many prime numbers. The proof is a standard part of a college number theory course, but can be understood by even a high school student and could be one part of a research paper on number theory. (An Internet search would quickly find this proof). Take another look at the list of prime numbers. You could make a list of things you notice. For example:

- There is only one even prime number.
- No primes are perfect squares.
- If a prime is equal to the sum of two other primes, one of the addends must be 2.
- The distances between consecutive primes do not strictly increase as the primes get larger.
- Lots of primes have 1, 3, 7 as their units' digit.

There are many more prime number patterns to be discovered. Look at the following pairs of primes, called **twin-prime pairs**:

$$3,5 \quad 5,7 \quad 11,13 \quad 17,19 \quad 29,31 \quad 41,43 \quad 59,61$$

Notice that we are able to list consecutive odd numbers that are prime to form twin-prime pairs. Can you add to this list? What patterns do you notice in the list of twin-prime pairs? Do you think there are infinitely many twin-prime pairs? The answer to this last question is unknown—it is an unsolved problem in mathematics! As you can see, unsolved problems do not necessarily have to be hard to understand, even though solving them may be challenging. Be aware that a list of thousands of twin-prime pairs, possibly generated on a computer, does not guarantee the existence of infinitely many twin-prime pairs. Such a list would certainly be helpful in making conjectures and looking for patterns, but it would not constitute a proof that there are infinitely many twin-prime pairs.

Do you think there is a formula that will generate only prime numbers? Examine the expression

$$y = x^2 - x + 41$$

Table 1.1 lists *x*-values from 0 to 41 and their corresponding *y*-values.

Table 1.1. Ordered Pairs Generated by the Formula $y = x^2 - x + 41$

x	**y**	**x**	**y**	**x**	**y**
0	41	14	223	28	797
1	41	15	251	29	853
2	43	16	281	30	911
3	47	17	313	31	971
4	53	18	347	32	1033
5	61	19	383	33	1097
6	71	20	421	34	1163
7	83	21	461	35	1231
8	97	22	503	36	1301
9	113	23	547	37	1373
10	131	24	593	38	1447
11	151	25	641	39	1523
12	173	26	691	40	1601
13	197	27	743	41	1681

You may have noticed that all of the numbers generated are odd. Will this always be true? If we factor part of the expression, we find that

$$y = x(x - 1) + 41$$

The product $x(x - 1)$ is always even, since x and $x - 1$ are consecutive integers. Do you know why? When 41 is added to this product, the sum value will be odd. The expression $x2 - x + 41$ is certainly an **odd-number generator**. Look at the first forty-one y-values generated by the formula (x = 0 to 40). These values are all prime. Based on the fact that the formula generated a prime the first forty-one times it was applied, would you believe that the formula is a prime number generator? Do you "trust" patterns that work for many examples and believe that they will *always* work? When $x = 41$ in this formula, $y = 1681$, a composite number. Therefore, the formula is not a prime number generator.

This result could have been found by trial and error with a calculator or by inspecting the formula and noticing that when 41 is substituted for x all three addends are multiples of 41, and as a result, the sum is divisible by 41 and is not prime. You may have noticed this from the start! Mathematicians have not yet found a prime number generator—this is another unsolved problem in mathematics.

Let's investigate our third unsolved problem regarding prime numbers. Examine all of the even integers greater than or equal to 4. See if you can express them as the sum of two prime numbers:

8 = 3 + 5

20 = 7 + 13

6 = 3 + 3

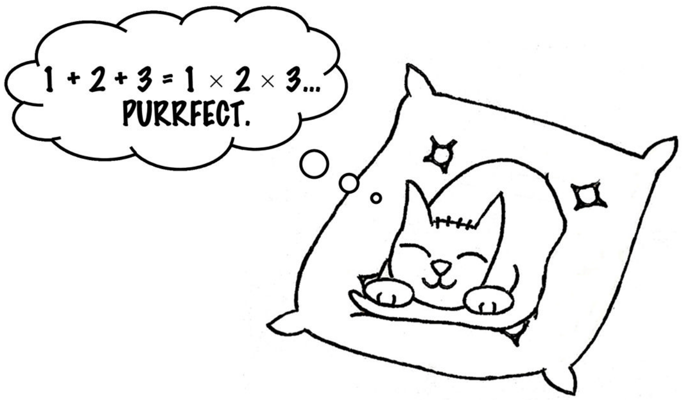
100 = 29 + 71

The statement "Every even integer greater than or equal to 4 can be expressed as the sum of two prime numbers" is known as **Goldbach's Conjecture**, and it is not known whether the statement is true or false. Can you tell why the theorem does not hold for all *odd* integers? (Hint: The only even prime number is 2.)

A **perfect number** is a positive integer that is equal to the sum of its proper factors. For example, the proper factors of 6 are 1, 2, and 3. Notice that

1 + 2 + 3 = 6.

Since 6 is the sum of its proper factors, it is a perfect number. Can you verify that 28 is the next perfect number after 6? It is unknown if there exists any odd perfect numbers. Also notice that the perfect number 6 is also the product of all of its proper factors, which makes it a special perfect number. Do any other perfect numbers have this property? Another unsolved problem in mathematics that can be explained to a fourth grader!



Perhaps you are surprised that such seemingly elementary topics as prime and perfect numbers has spawned such frustration. But there are rewards. The August 8, 2002, *New York Times* reported that three Indian computer scientists had conquered an unsolved problem involving prime numbers. They devised an efficient way for a computer to determine whether a number is prime. To learn more about prime number challenges that have been answered, read Peter Plichta's book *God's Secret Formula: Deciphering the Riddle of the Universe and the Prime Number Code* (1997).

Of course, all "solved" problems started out as unsolved problems. It could take minutes, or centuries, to find a proofs for a particular conjecture. There are many different proofs of the Pythagorean theorem,

$$x^2 + y^2 = z^2.$$

When the solutions are all positive integers, the solution set is called a **Pythagorean triple**. For example, 3, 4, 5 and 8, 15, 17 are Pythagorean triples. If the exponent '2' was replaced with '3,' a new equation is formed:

$$x^3 + y^3 = z^3.$$

Does this equation have any solutions that are all positive integers? Let n be a positive integer. Examine this equation:

$$x^n + y^n = z^n.$$

Do equations of this form *ever* have positive integer solutions? In 1637, Pierre de Fermat conjectured that they did not; that no integer solutions exist when $n > 2$. He claimed he had a proof, but it was never found. Over 350 years later, somebody finally proved what became known as **Fermat's Last Theorem**! It took three and a half centuries of trials, persistence and creativity to prove this theorem—it was not done in the confines of a 45-minute math class!

Sometimes school can give students the impression that all of math is already discovered, and the teacher has all the answers. Nothing could be further from the truth! Many of the current questions in mathematics will not be answered in your lifetime, but perhaps some of them will. The attempts will require persistence and creativity. The ability to persevere is often accompanied by frustration from errant attempts. This frustration is then harnessed to provide the energy for the next attempt.

Questions and Answers—Fun and Frustration

Ever wonder why people like puzzles and mathematical brainteasers? They epitomize fun and frustration. Fun and frustration?! That might sound oxymoronic, but the proliferation and popularity of mathematics and logic games, puzzles and periodicals assures us that people enjoy challenges—that fun and frustration *can* go hand-in-hand.

You might be surprised to know that one of the most famous television cartoons of all time, *The Simpsons*, is written by mathematicians, who have embedded cleverly hidden mathematical surprises in many episodes. *The Simpsons and Their Mathematical Secrets* by Simon Singh (2013) is an excellent read for anyone interested in a fun look at the beauty and spirit of mathematics.

> *"The true spirit of delight, the exaltations, the sense of being more than man, which is the touchstone of excellence, is to be found in mathematics as surely as in poetry."*
>
> —Bertrand Russell, *A History of Western Philosophy,* 1945

Throughout this chapter, you "heard" mathematics shout questions. The many questions offered here comprise just the tip of the iceberg. The topic you choose will be loaded with questions. Chapter 2 will help you find a topic. Because mathematics only whispers answers, Chapter 3 focuses on problem solving, which will help you "hear" answers. The thrill of experimentation and discovery awaits!

Chapter 2

Finding a Topic

At this juncture, it would be a good idea to find a topic for your research paper, so you can get an early start. Chapter 2, along with the resources in Chapter 12, will help you find a topic. Chapters 3-7 discuss the role of questioning, problem solving, writing, and proofs in mathematics. Chapters 8-12 will direct you through your research project. You have probably written many reports during your education, but you may never have done a research project. How does a research paper differ from a report?

Reports and Research Papers

When an instructor assigns a report, most often it involves your going to a library or using the Internet. Reference materials from a library or the Internet often provide all the information necessary to construct a report. The material you gather and read was probably written by experts in the field. A report on the Civil War can include any information on the Civil War. You select the material you will include, organize it, write an outline, and write your paper. You proofread and revise your paper several times to make it as professional as possible. Writing a report is often a good way to orient yourself to a topic you are unfamiliar with. Reading a student's paper on the Civil War might be a good way for a tourist from another country to get an encapsulated introduction to the Civil War. As the author of a report, you are a *reporter*.

As a *researcher,* you start with a problem—a question that needs to be addressed. You read related material to try to formulate an answer to the question. You may have to do an experiment or compile evidence to support the solution you present. Original material can be added where necessary to help solve the problem. You generate other questions based on your readings and the tinkering you've done with the problem. Only information that contributes to the solution of the problem is included in the main body of the research paper. Material tangential to the narrow focus of the research problem is not included in the main body. Some of these tangents might be recommendations of topics for other researchers to study. You need to stay focused on explaining aspects of your central problem. You become a specialist on your problem. You are a participant in the creation of material for the research paper. At times, you may get stuck and need assistance in attacking a part of your solution. You might consult a mentor, an expert in the field, or your library, or you may post a message in an appropriate chat room or on a listserv message board. Throughout your research project, the central focus is the solution of a specific question or set of questions.

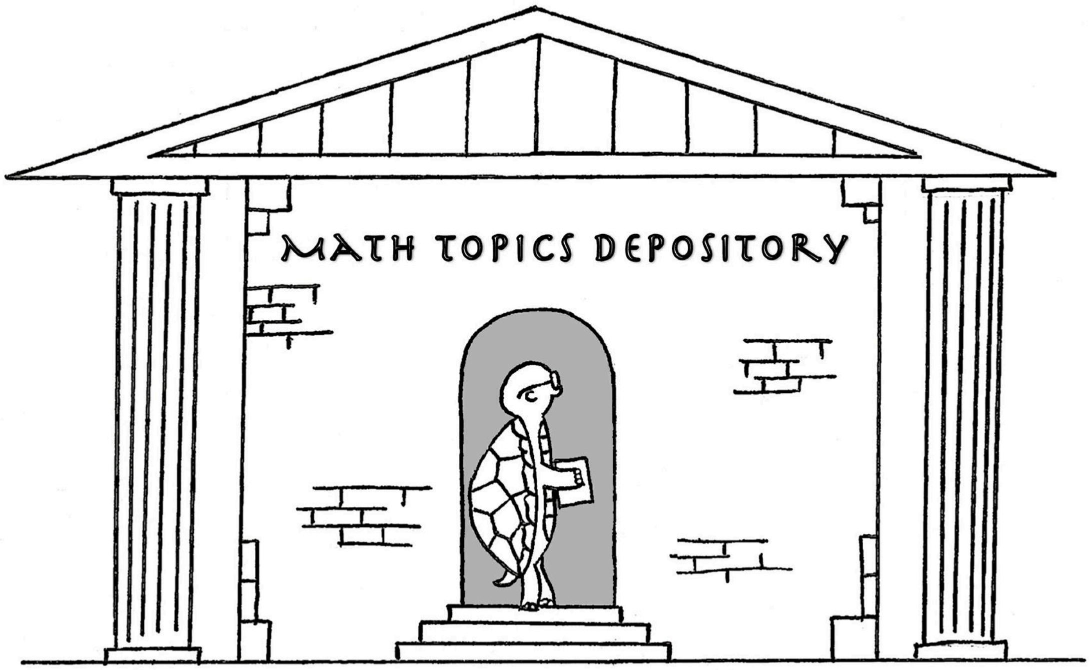
Focusing on one problem tends to keep research papers very specific. A student could write a math report on the Fibonacci sequence, but this would be too broad for a math research paper. "The Use of Fibonacci Numbers to Create Pythagorean Triples" is a specific topic that could be the focus of a math research paper. Topics such as "Probability," "Geometry," or "The Pythagorean Theorem" are too broad. The following are examples of topics that have a specific focus and would make suitable research paper topics:

- Determining the conditions necessary for a quadrilateral to be able to be both inscribed in a circle and circumscribed around a circle
- How Fibonacci numbers can be used to find Pythagorean triples
- Using probability to find the area of irregular figures
- Using the area of a circle to find the area of an ellipse
- Finding maximum and minimum function values without using calculus
- Solving quadratic equations using a compass
- Reflections in a cylindrical mirror
- How an addition table can be used to find Pythagorean triples
- Finding the ratio of the radii of the inscribed circle to the circumscribed circle of a right triangle with integer-length sides
- Determining when quadratic equations with consecutive integer coefficients can be factored
- Finding the number of regions created when a given number of chords are drawn in a circle
- Determining if an angular cut through a cylinder forms an ellipse
- Finding a minimum length path along a surface of a solid
- Determining the shortest path between two points on a cylinder
- How a cubic equation's roots can be found using a straightedge
- Finding a quadrilateral with numerically equivalent area and perimeter, that has side lengths which are natural numbers

As you can see, the title of a research paper has the potential to be long because it must be descriptive. Chapter 9 discusses titles and other components of the formal paper in detail. Your main concern here is to pick a topic with a narrow scope. As a newcomer to math research, you can either create your own problem or question to research or use journal articles to provide the problem for your paper. Each article solves a specific problem and offers information essential to the solution of that problem. Using a journal article will help you pick a focused topic. Other factors that should guide your choice of a topic are discussed in the next section.

Choosing Your Research Project

Finding a topic is the first major milestone of your research project. If you have the option of choosing your own topic, you have a responsibility to yourself. You will be spending extensive time working with the topic, so you want to pick a stimulating topic that is appropriate for your background. How do you find a topic for your math research paper?



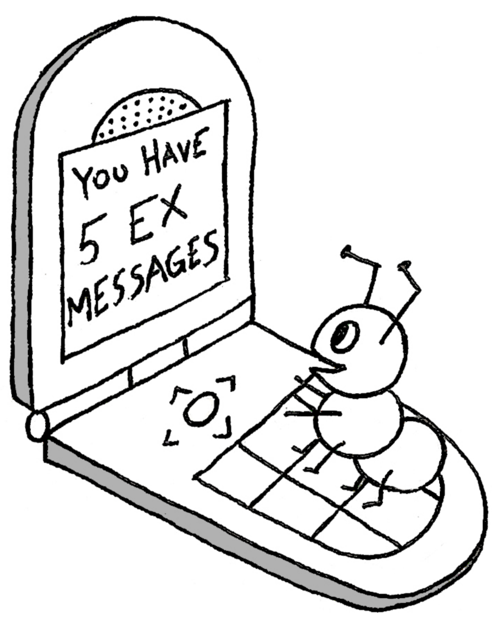
When you have a choice as to what research project to tackle, remember your investment in the project and consider the five "Ex" messages as you make your choice:

> **EXPERIENCE:** You should have some familiarity with some of the prerequisites necessary to tackle your topic; it should not be 100% foreign to you, and/or miles above what you've already studied. You'd like to be able to *use* some of the math you've learned to tackle this new topic. The topic you choose might even be an outgrowth of a topic you enjoyed in your regular math course. Often, advanced, difficult topics entrap you into producing a report, not a research paper. You need mathematical "dexterity" to do research.
>
> **EXCITEMENT:** You should *like* your topic—make sure you think it's cool. Don't just pick a topic by title from a grocery list. Your experience with concepts related to your topic will help. Don't pick a topic just because it sounds "hard" or impressive.

EXPERTISE: People research topics they have been successful with facets of already. Nobody just wakes up one day and decides to arbitrarily research a random topic. People who are invested in their field always want to use their expertise to further knowledge in that field. Posing questions, testing and forming conjectures, and reading about extensions of the topic rely on a knowledge base.

EXPLORATION: It is nice to find supplementary materials once you are considering a topic. Look in the bibliography of whatever got you initially interested in your topic. Consider getting some of the related articles that were cited in the bibliography. You can "Google" your topic and explore sites that might help you with what you are doing. Consider getting a textbook, which would do a good job of teaching you some prerequisites necessary for your study.

EXTENSION: Think of ways you can investigate some aspect of your topic that was not in your readings by extending it into "new" territory. If you pick a topic that is too far above your knowledge base, it might be too difficult for you to extend it. You want to be able to creatively "play" with mathematical findings that are new to you. Keep in mind that your extensions will arise as you *do* your research; they won't be apparent to you when you *pick* your topic.



Researchers who choose their own topics naturally pick extensions of work they are entrenched in. For example, a nutritionist doing research would naturally be interested in the effects of cholesterol on the human heart. An aircraft designer would naturally be interested in researching the advantages of a new wing design on commercial air flights. Researchers can be fanatical about their work—their dedication and obsession are by-products of a long-term commitment to their field. As a newcomer to research, how do you fit in?

Your Mathematical Experience

Because much of the territory you chart with your research will initially be foreign to you, familiarity with a topic is important in your choice of a research topic. Examine this scenario:

You take the same route walking to class every day. A student you don't know passes you in the hall every day. You don't say hello to each other; you barely exchange glances. You don't know each other's names. Years go by, and this relationship stays the same. One summer, you go on a vacation to Paris. You get off the plane, and standing there in the airport in Paris is the same student you passed on your way to class! Do you walk up and say "Hello" *now?* You bet!

In foreign surroundings, we all take comfort in a bit of familiarity. What does this have to do with your research topic? In the same way that you find comfort in a familiar face in unfamiliar surroundings, you will use mathematics you are familiar with as an anchor as you proceed through your research. As you learn new concepts, they will then become familiar and become additional anchors.

As a student, you have sat in math classes for over a decade, and the best is yet to come. High school curricula and college programs are packed with many thought-provoking, challenging units. Do you have any general preferences regarding mathematics at this time?

- Would you prefer researching pure mathematics or applied mathematics?
- Are you a whiz at algebra?
- Do you enjoy geometry because of its visual nature?
- Do you wonder how different types of graphs are related to their equations?
- Are you fascinated by number theory?

Perhaps you have already wondered about some specific mathematical ideas that you have encountered in your education or in your daily life.

- Do you wonder how satellite dish antennas are engineered?
- What do you know about the Pythagorean theorem besides $a^2 + b^2 = c^2$?
- How is the graph of a parabola related to the quadratic formula?
- Can you imagine the Fibonacci sequence having some applications?
- How is trigonometry used to find distances in our solar system?
- How can graphs be used to find how a corporation can maximize its profit?

The possibilities are endless. As we mentioned previously, research topics must be narrow—they must be focused. Picking a topic that is too broad is a common mistake students make when they are writing a mathematics paper without sufficient guidance. Students who pick a broad topic often find many library books on the topic, as if they were writing a report. They then try to cull different pieces of information from the books to come up with a gelled report. Having many sources is not necessarily an advantage in writing a math research paper. A lengthy bibliography does not make the paper stronger. It is actually very difficult to select and integrate material from several very comprehensive books and then write a paper. It is easier to internalize work that comes from *you*. Classically, people start researching because one specific problem is gnawing at them, not because they got the urge to write a synopsis of several thousand pages of others' work. A mathematician could spend an entire lifetime researching one particular problem in probability; writing a single paper on probability would be trying to cover too much. Your paper will not be a summary of the material in many books. Quite the opposite—it may be built out of only a few pages and your ability to inquire about, process, and explain mathematical thoughts. Given the role of mathematical experience, how can you find a focused topic if your mathematics background is somewhat limited? The following sections give you some suggestions.

Building on Short Articles

My recommendation for finding a focused topic is to use an article from a mathematics book, a mathematics journal, or a math education journal. Look in Chapter 13 for information on books of short articles that can form the basis of a research paper, as well as several popular journals. Additionally, your state's mathematics teachers association may publish a journal; see your math instructor or the department chairperson for this information. The journals are kept in the periodicals section of the library. Many can be accessed through the Internet. Older copies are available on microfilm, microfiche, or the Internet, and more recent issues may be found bound together by year or as separate publications. Your math department may have a collection of back issues of certain journals or a collection of photocopied journal articles for you to look at. Check your library. Plan to spend some time looking through the journals once you've found them.

In each issue, the journals feature articles less than ten pages long (usually three to six pages) on a specific concept in pure or applied mathematics. Often they describe some novel aspect of a common topic or provide a new twist to an old topic in a succinct yet comprehensive way. As you read the article, you will get an education on some new aspect of mathematics you have not met before. This is another advantage to using journal articles—you get to learn a bit about a new topic, and then you get to expand on it. By their nature, these articles are already narrow in scope. Thus, starting with a journal article provides the focus you need to write a successful paper. Plus, that narrow scope gives you room to try your own extensions when you have mastered the mathematics in your article.

As a result, it is likely that the bibliography for your paper will consist of only a few items—the article that sets you in motion, and some other related sources you found as you explored the Internet and the library. Chapter 8 will explain why and how a six-page article can provide you with months' worth of work. Your objective in Chapter 7 is to find a topic for your research project.

Take inventory of the mathematics that interests you. There will be only a few articles in each issue of a journal. Look at the tables of contents. Plan to look over all the articles; try not to rule out articles solely on the basis of their title. Carefully read the first paragraph or two of articles that look interesting, and then skim through the rest of them. Don't expect to understand everything. Try to determine which articles are reasonable extensions of your current knowledge. Pick your favorite three articles in order of preference. Show the articles to your instructor. He or she can help you determine a suitable choice given your math background. Don't attempt something that is loaded with prerequisites with which you are unfamiliar. Follow your instructor's advice. Once you've chosen an article upon which to base your research, you may need to find other sources to support your research. You can also ask your instructor for recommendations of textbooks and internet sites you can use to learn the prerequisite skills the article assumes the reader has. There are many videos on the Internet which can be used to learn these prerequisite skills. Since you can watch them over and over, you have the chance to really try and grasp the new information on your own. This is another huge building block of research—being able to conquer new territory without asking for help.

These accounts of actual students' research will give you ideas for how students learned the prerequisite skills they needed in order to read and understand their articles.

- Daniele was working with probability and the quadratic equation. She needed experience with conic sections—specifically, the parabola. Daniele's first research task was mastering the graphs, equations, and properties of parabolas.
- Jeff's research on special cases of similar polygons required a knowledge of proportions in the right triangle. An old geometry textbook provided all the information Jeff needed.
- Simran's original proofs of properties of skewsquares required familiarity with proportions in similar triangles. These proportions were explained in an integrated mathematics textbook.
- Midway through Luke's research on the areas of irregular plane figures, he needed to learn about the trapezoidal rule, a topic from calculus. Luke looked online and found a two-page treatment of the trapezoidal rule, and studied it before continuing with his research.
- Katie's work on number theory required her to learn about numbers in bases other than ten. The article was heavily dependent on mastery of this concept. She reviewed information on other bases in a middle-school textbook.
- Part of Robin's work on a famous problem involving triangles and circles required that she learn partial derivatives and optimization theory. This is an

example of an article with too many difficult prerequisites, and as a result this part of the article was not researched. The first part of the article provided her with excellent material on which to build a paper.

If you set aside a few hours, you can look through dozens of articles. Hopefully, the suggestions for finding sources and articles given above will get you started. Keep in mind that you may not be able to finish researching your article in your current school term. On the other hand, you might have time to read several articles on related topics and combine the results in your paper. Also remember that you need to pick an article appropriate for your background. For example, a high school student interested in the effects of high blood pressure on sleep disorders in children cannot expect to do a research paper on that topic that is as comprehensive as one written by a doctor who has specialized in sleep disorders for the past twenty years.

After you have chosen the article you plan to use, make two or three photocopies of it. You will want to work with photocopies because you do not want to mark up an original copy of an article. You will need an archive copy, which stays at home in case your other copies are lost, and an annotation copy, which you work on. You may decide to read several related articles and combine some of the findings to create a new result. Make copies of all the articles you intend to use. Use the terms presented in the article as key descriptors to do Internet searches to find related material on your topic.

Other Ideas for Research Topics

Journal articles are highly recommended as sources for your research topic because they provide direction in a succinct, comprehensive way well suited to new researchers. In this section, we discuss other sources for a topic. Keep in mind that these sources will not provide as much written material as journal articles; consequently, from the outset, you will be doing lots of probing on your own. Here are some suggestions:

Use a problem as a springboard. In your problem-solving work, you may encounter a problem that is only one-to-four sentences long that really intrigues you. After you find a solution, ideas may fill your head as to how the problem could be altered slightly, extended, or changed radically to provide new challenges. The new problems you pose and their solutions and/or proofs could comprise a research paper totally inspired by that one problem. Read the following problem:

> Rich has devised a carnival game. He has a large square board tessellated with 2500 one-inch squares, in a 50-by-50 pattern. The squares are red or black and are in a checkerboard pattern. A person who wants to play the game tosses a dime on the board. If the dime lands totally within a red square, the player wins. If it lands in a black square or overlaps several squares, the player loses the dime. What is the player's probability of winning?

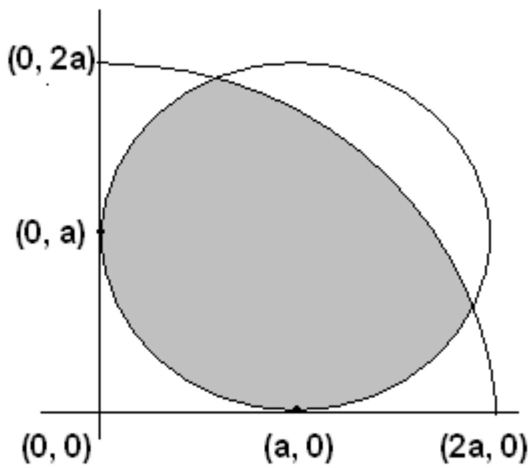
A common solution involves solving a simpler problem. Think of all the questions that might arise.

- What is the diameter of a dime?
- What if there were only 100 squares?
- What if the game used quarters?
- What if the board used triangles?
- What if the board used hexagons?
- Which polygons do and don't tessellate? Why?
- What if all of the squares were red, with black lines dividing them into 2500 squares?
- What if the board contained infinitely many squares?
- Could you pretend that the board has just one square and find the solution to that problem, and then adjust it for the 2500 squares?
- How would the theoretical probability change if the "board" was three- dimensional and the player threw a sphere instead of a circle? The original problem (appropriately footnoted), its solution, and your conjectures, extensions, and so on can make a great paper. You must have access to someone who can check the accuracy of your work frequently. Chapters 12 and 13 list many problem-solving resources.

Use sources from the Internet. There are many Web sites that state problems suggested for research. Many of these sites link to other sites, and some even offer help to students. If you have some topic names to start with, you can do searches of your own to find research topics. Keep in mind that you need *some* direction and focus to do an Internet search; you can't just randomly find a topic.

Use another student's recommendations. You may watch an oral presentation from another student as part of your math research class. Research papers don't end with a "summary;" they end with a section on recommendations for further research. This sections poses new problems based on the completed research. You may decide to do research on one of the recommendations made in another student's research paper.

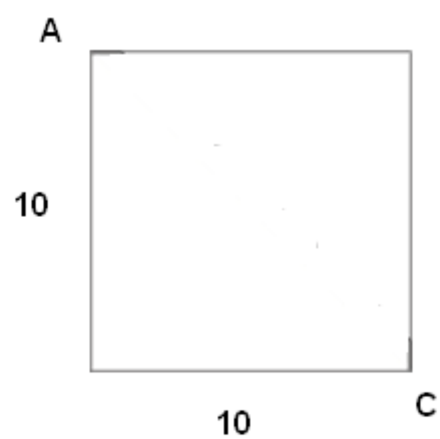
Think of a research problem on your own. As a person attuned to mathematics, you might possibly have a question hit you and decide to research it. Blake had such an experience. He wondered how he would find the area of the shaded intersection (he named it a "bicusp") of a quarter circle with a radius equal to the diameter of an intersecting circle. He wanted to create an original formula. The circles are situated as shown in this figure. The quarter circle has radius 2a, and the circle has diameter 2a.



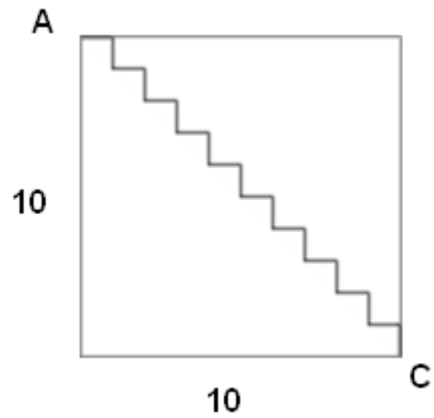
Blake's original conjecture involved an erroneous assumption about proportions, but finding the area proved to be a terrific exercise in analytic geometry, trigonometry, and calculus. The entire research paper was based on a query Blake had and on no other written inspiration. He did, however, use textbooks to learn concepts he needed in order to continue his work. If you create your own problem, you need to be very industrious and motivated to find the answer.

Linda once created her own problem based on a simple problem involving the Pythagorean theorem and distance.

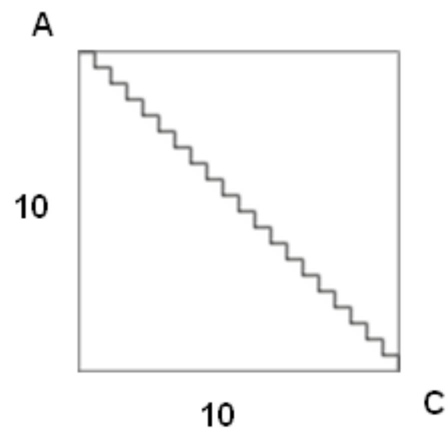
In the following square *ABCD,* the distance from *A* to *C,* if you walk along the square, is 20.



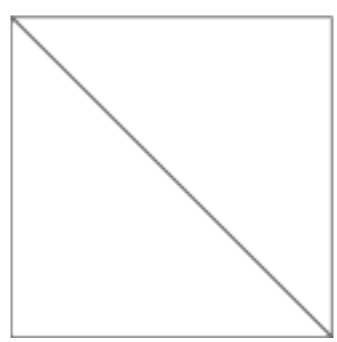
If you walk along the "steps" from *A* to *C* in any of the following diagrams, the distance is still 20.



If the rise and run of the steps becomes very small, the distance from *A* to *C* is still 20.



If you walk along a straight line (the diagonal) from *A* to *C*, the distance is approximately 14.14.



What happened to make this distance from A to C “jump” from 20 to 14.14? When Linda pursued the solution, she found that it had many prerequisites in calculus, which she had not yet taken. She decided to find another research topic and save this one for a later day.

Extend an idea from your core mathematics course. As you engage daily in a core mathematics course, you have a chance to probe, conjecture on, and wonder about variations on what you are learning. You may decide to research an extension of a certain topic from your regular math class.

Laura studied quadratic functions in her algebra class. She also had a brief introduction to simple absolute-value functions. Laura wondered about the graphs of quadratic functions that contain absolute-value expressions, such as the following:

$$y = |x^2 - 5x - 6|$$

$$y = |x^2 - 5x - 6| + 12$$

$$y = |x^2 - 5x - 6| + 4x - 9$$

$$y = |x^2 - 5x - 6| - 3x^2 - 5x - 11$$

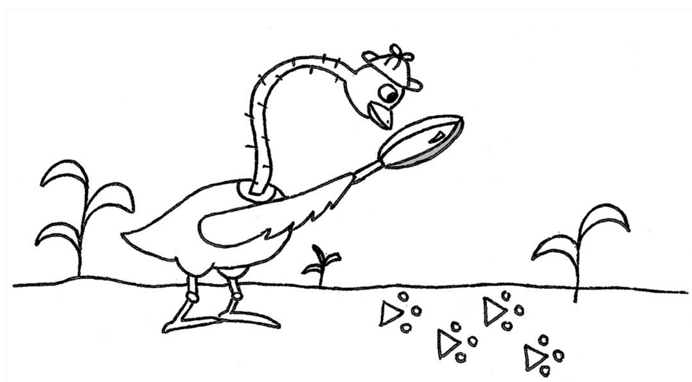
Laura’s research on these graphs involved a computer, a graphing calculator, and pencil and paper. Her entire exploration was derived from topics covered in her regular math class. She extended the topic by addressing the parts she “wondered” about that were not part of the curriculum. You can think about extensions to every topic you learn in your core math class.

Use an application from real life. You might be curious about a mathematically influenced aspect of the real world. Students have wondered about the shape of airplane wings, how skyscrapers are engineered, why honeycombs are based on the hexagon, how codes have been used in wartime, how suspension bridges are built, and how interest is compounded, to name just a few topics. Mathematics is used in music, medicine, sports, astronomy, and many other fields, and these fields have inspired many math research papers. For example, Gino had often wondered why car radio antennas had a practical shape, a telescoping mast, while satellite dish antennas seemed so large and clumsy. He wondered about the shape of the satellite antenna. His research led him to journal articles about these antennas. He also needed some material from a textbook and from his math course, because the antenna’s shape is based on conic sections. In addition to reading these articles, Gino contacted companies that manufactured and/or used the large antennas. Similarly, Pam wondered about space telescopes and contact the University of Arizona about their GMT—Giant Magellan Telescope—and received lots of helpful information.

Excellent application problems abound in mathematics. Applications can make good research topics, but be forewarned that you can easily be led into other disciplines, such as science or engineering, that may require years of background for you to fully understand a particular research topic. You can find more information about writing a math application research paper in Chapter 10. On your own, you are more likely to be able to find new patterns in and experiment with the Pythagorean theorem, the Fibonacci sequence, and so on, than to improve upon the design of the Golden Gate Bridge. Keep this in mind as you choose your topic.

Getting Started

You have been introduced to several ways of finding a topic for your math research paper. Allocate a reasonable amount of time to find out where you can locate the materials described in Chapter 12. Keep in mind that the articles provide more focus than the other sources mentioned in this chapter. Get your copies and start on your topic as soon as you finish reading this chapter. You will be working on your research and reading *Writing Math Research Papers* concurrently. Refer to chapters in the book, the time line in the Introduction, and your instructor if you need assistance.



Making a poor topic choice will be a tremendous hardship as your paper progresses. Use the advice in this chapter and some of the suggestions in Chapter 12 before you decide on a topic. This book's advice is based on the experiences of hundreds of students who have written math research papers over the past two-plus decades—virtually all of them with great success, because they followed the suggestions in *Writing Math Research Papers* carefully. Remember—research, by its nature involve uncharted territories, so do expect some bumps along the way. Bumps can be frustrating, but they are very rewarding to overcome! It is important to pick your topic carefully. These stories illustrate some of the difficulties you'll want to avoid when picking a topic:

- Joelle was interested in optical illusions and decided to do a paper on them. She found many sources, but hardly any that discussed the illusions mathematically, so she had trouble extending her topic with original material.

- Adam was a musician who decided to do a paper on math in music. He found articles that either were too simple or were beyond his ability, but no articles between these levels of difficulty.

- Nicole wanted to do her paper on the mathematics of aerodynamics in flight. However, much of the material she found required some background knowledge of physics that she didn't have. Eventually, she had to change her topic.

Try to stick with a topic once you have found one. If you change topics, the time you have invested in the original topic will be lost, and you will have less time left in the school year to complete work on another topic. That's why it is best to use care, discretion, and instructor guidance when you are initially choosing your topic. It is also wise to make your copies and begin your reading immediately. As you do your research, you will generate your own questions and try to answer them. You will be actively engaged in the research. Because you will not formally write your research paper until your research is well under way, you will keep a journal of notes about your readings and findings. Chapters, 4-8 will assist you in keeping your research journal and writing up your findings.

Chapter 3

Problem Solving: A Prerequisite for Research

> *Solving problems is a practical art, like swimming, or skiing, or playing the piano: you can learn it only by imitation and practice . . . if you wish to learn swimming you have to go into the water, and if you wish to become a problem solver you have to solve problems.*
>
> —George Polya, *Mathematical Discovery,* 1961

As the world becomes politically, economically, and technologically more complex, problem solving has become a primary focus of schools, government, and businesses. The rote skills taught to students in years past would not have equipped them to solve the problems facing the world today. Think of some differences between today's world and the world of fifty years ago, and discuss answers to the following questions with your classmates before reading further.

- What is problem solving?
- Why is problem solving a sensible focus for schools and colleges?
- What role do you think problem solving plays in math research?
- Why do businesses and the government need problem solvers?
- Why is it so difficult to solve global problems?

Many of the problems facing the world today are new, non-routine problems. How will you handle the problems you encounter? Is there a *routine* for researching the solutions to *non-routine* questions? You won't be able to simply mimic old solutions. You will face problems that haven't even been conceived of yet, so the problem-solving strategies you learn must be extendible. The strategies and tactics you study cannot consist of mere facts— they must be *procedures* that can be used in all or most problem-solving situations. As a discipline, problem solving is a collection of strategies that can be used to find a solution to a specific situation called the **problem.** These problem-solving strategies apply to all situations, not just mathematics.

This chapter serves as an abbreviated refresher for people who have problem-solving experience and as an overview for people new to the problem-solving arena. If you are an experienced problem solver, you know that the accumulation of strategies, tips, and pitfalls will strengthen your skills. This chapter will benefit you in that sense as well. An excellent, thorough introductory treatment of problem solving can be found in *Problem Solving Strategies—Crossing the River with Dogs,* by Ted Herr and Ken Johnson. George Polya's *How to Solve It* is also a must-read for the serious problem solver. These and other books on problem solving are listed in Chapter 13. Look for some of these books in your library; they will help you sharpen your problem-solving skills.

The skills you acquire in a problem-solving course will have a direct carryover to your mathematics research. You may have thought of a mathematics research paper as a "math book report." As you read through this book, however, you will deepen your understanding of the difference between writing a report and doing research. You will have a firsthand look at the interdependence of problem solving and math research. Your research will lead you to many non-routine situations; persistence and problem-solving skills will optimize your chances of arriving at a solution. Throughout your research, you will encounter claims and statements that you don't understand. You will discover patterns and formulate conjectures that you want to prove or find counterexamples for. You will often be in the position of trying to decipher and master something you have never seen before. The problems that form the cornerstone of a problem-solving curriculum are small capsules of the hurdles you'll meet as your research progresses. The strategies used in problem solving are effective strategies for attacking these obstacles, because they chart a systematic path through uncharted territory.

Problem solving is inherent to all branches of mathematics. The research and problem-solving skills you acquire through mathematics will help you solve not only purely mathematical and math-related problems but also non-math problems, because the essence of problem solving is a logical, systematic process that leads to a solution. An analogy can be made to the changing of a flat tire. A road service can be called to the scene of a flat tire, and a mechanic can change the tire for the driver who does not know how to do it. The driver's problem is solved. The very same mechanic can teach the driver about the tools and safety procedures involved in changing a flat tire. Equipped with these extendible skills, the driver can safely change flats in future roadside emergencies without assistance and on different cars. In addition to having the problem solved, the driver has learned *how* to solve the problem. Learning strategies is more effective than merely receiving answers, because strategies can be extended, but an answer is applicable only to one specific question.

Our Quantitatively Oriented World

Take a look around at the technological world you live in. Look at the new discoveries made in your lifetime alone. Look at how the workplace is changing. The demands of the twenty-first century require a citizenry well versed in problem solving and in meeting non-routine, unanticipated demands both at home and in the workplace. Imagine these scenarios:

- A city is planning to construct a domed stadium. You are in charge of the construction of the dome. You have many factors to consider. How much weight must the dome support? How much do snow and rain weigh? How will wind affect the dome? How will expansion and contraction due to temperature changes affect the dome? What are the least expensive materials that will meet your demands for strength? How does the probability of more severe, but rare, stress and weight circumstances affect your decision as to whether to build a stronger, but more costly, dome? In answering these questions, you will face a constant battle of cost versus quality of materials and workmanship. Will your dome stay structurally sound after a 10-foot snowfall? What is the probability of such a storm in your area? The solutions to these scientific, economic, and safety issues all incorporate mathematics.
- You are a consultant to a team of scientists that is on the verge of finding a cure for a major disease. The cure involves securing materials from tropical rain forests. Environmental, political, health, cultural, and sociological concerns are involved. The debate is laced with mathematics. How much of the medicine is needed? How much of the rain forest needs to be depleted? How will this depletion impact the environment? How long will it take to renew the depleted acreage? How will the medicine affect the life span of ill individuals? What impact will the increased life span of afflicted individuals have on available medical facilities?
- Your company has created an automobile engine that lasts four times longer than the current breed of engines without needing any repairs. How much more will the engine cost the consumer? How drastically will it affect the auto repair business? What changes will need to be made to the rest of the car so it can last long enough to take advantage of extended engine life? The domino effect on automobile manufacturers and related industries has a major impact on the economy of the entire nation.

Try to think of your own scenarios that require non-routine problem solving in our increasingly quantitative-oriented society.

The Routine vs. the Non-Routine

Have you ever tried to do a homework problem, encountered difficulty, and said to yourself, "We never did one like that in class!" Maybe you thought it was unfair to be given such an assignment. Schools and colleges provide you with problem-solving training by exposing you to many problem situations in your classes. You must learn how to extend and apply, with discretion, the tremendous amount of knowledge you have amassed during your education.

Let's use an example to show the difference between doing a typical homework example and solving a problem.

When you learn how to factor trinomials, you might drill yourself by doing five, ten, or 100 examples and, through this practice, become adept in this one skill. We wouldn't usually consider the 101st trinomial factorization you try, say, $x^2 - 4x + 5$, a "problem." The first one you try, if you factor it on your own, without benefit of an instructor's lesson, might be considered a problem. Though small in scope, it may represent a new, non-routine situation for you for which you employ some problem-solving strategies and content knowledge to arrive at an answer. *Could* the 101st trinomial factorization be considered a problem? Yes, if it involves a new situation. If the coefficients make finding a correct factorization less routine than it was in the previous examples, it is a problem. Look at the example below:

$$\text{Factor: } 36x^2 - 255x + 100$$

The large number of combinations of factors of 36 and 100 may require you to do some groundbreaking above and beyond the skills required to factor $x^2 - 8x + 15$. In that sense, this factoring question represents a non-routine situation. (Try to factor it!)

Problem solving classically involves non-routine situations, such as the following example, whose solution depends on the content knowledge of factoring.

Problem: Prove that if x is a positive integer, then $5x^3 - 5x$ is always divisible by 30.

Solution: For $5x^3 - 5x$ to be divisible by 30, the expression $5(x^3 - x)$ would also have to be divisible by 30. This would mean that $x^3 - x$ would have to be divisible by 6. The expression $x^3 - x$ can be factored as follows:

$$x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1)$$

These three factors can be commuted:

$$(x - 1)x(x + 1)$$

Notice that if x is an integer, the product above is the product of three consecutive integers. The product of any three consecutive integers is always divisible by 3, because one of the integers must be a multiple of 3. Also, at least one of the integers must be even, that is, divisible by 2. As a result, $x^3 - x$ is divisible by 6. We can now state that the difference between a positive number and its cube is always divisible by 6. Consequently, $5x^3 - 5x$ is always divisible by 30.

Do you consider the problem and its solution routine? Would you have solved the problem in the same amount of time it took you to factor a dozen trinomials? A contradiction arises. How can the non-routine become routine? How can anybody, ever, become a good problem solver if, in fact, the next problem he or she meets does not mimic one already seen? The answer to this question lies in the chapter-opening quote by Polya and in the fact that when you learn how to problem-solve you are learning *approaches* to solving problems. "Problem solving has been defined as what to do when you don't know what to do" (Herr & Johnson, 1994, p. 2). If you learn a number of approaches and practice them on enough problems, you'll learn how to apply these approaches to new situations.

A General Approach to Problem Solving

George Polya's classic book on problem solving, *How to Solve It,* was first published in 1945. Many decades later, the timelessness of the book remains a credit to Polya's practical approach to solving problems. You should find this book in your library and take the time to read it. In the book, Polya describes four steps critical to the solution of any problem:

1. Understand the problem.

2. Devise a plan to solve the problem.

3. Carry out the plan.

4. Look back and check the result.

A well-organized problem-solving strategy is a superb research tool. We will use the acronym SUPERB to embellish and explain Polya's problem-solving steps and make them easy to remember.

S: Scrutinize the problem. Read the entire problem thoroughly to the end. Don't skim. Don't make assumptions. Be open-minded. Begin by mentally sorting out the given information. Note if too much information is given or not enough.

U: Underline key phrases, questions, requirements, conditions, and so on. In problem solving and math research, it is always wise to have a colored highlighter on hand and to use photocopies rather than original sources so you can write on the problem. Draw key diagrams. Express key phrases in math notation.

P: Plan an appropriate problem-solving strategy. You may know some already. Several problem-solving strategies will be discussed later, and you may plan to use one strategy or a combination. Don't worry about specific data; simply formulate a general outline of what you will do to solve the problem.

E: Execute your plan. Carry out the procedure you set up for your solution, and find the solution. Save all of your work, even the dead-ends. Try to be organized, so that if you stop work in the middle of a process, you'll understand your previous work when you return to it. It will also be easier to check the accuracy of each step you took if your work is organized. Often, problem-solving attempts are done in a scratch-work style, and that is fine, but make sure you can decipher what you have done! If at some point the work must be made comprehensible to an outside reader, your ability to write and explain the solution becomes critical. The solution must be written in complete sentences.

R: Review the work you've done. Check to see if your answer makes sense. Make sure you answered the question in full sentences. Review your steps and any arithmetic or algorithms for correctness. Check that you applied the correct data. Determine whether your solution contradicts any of the conditions of the problem.

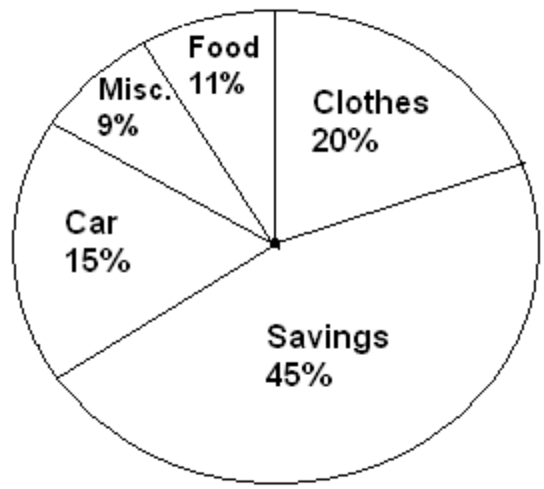
B: Build this solution into your problem-solving arsenal by writing out all the steps of the solution in full sentences. Write out the problem-solving strategies you used. This will help you sharpen your writing skills for your research and give you practice in expressing your findings so they can be understood by others. It will also help you recall how you solved the problem if you return to it in the future to help solve a related problem. List a few questions that come to mind about the problem, or create a new question based on the problem. Write clearly! Something that seems obvious to you as you discuss the problem with classmates now may not be so obvious weeks from now.

Why Scrutinize?

The "scrutinize" step augments Polya's "understand the problem" step. It is designed as a precaution. Too often, good mathematics students attack their schoolwork, especially timed exercises, with a voracious appetite. While admirable, this often results in their starting problems before reading them carefully, anticipating the question that will be asked before it is asked, and attempting a solution without relaxing, focusing, and rereading the problem. Try not to be too anxious when you approach a problem. Reading a problem is different from reading a magazine article. Additionally, not all problems come from books. Some come from real-life scenarios, even your own experiences, and might include extraneous information, not enough information, or incorrect information. The "scrutinize" step is a reminder to focus on relevant information in each problem. Scrutinize and then solve the following problem.

Jacquie's monthly income of $1460 is allocated as shown in the pie chart.

How much of Jacquie's income goes towards her greatest expense?



A superficial look at the problem would lead you to take 45% of $1460 and report that the answer is $657. A closer look might cause you to ask, Do we consider savings to be an *expense?* If not, clothes are her greatest expense, and the answer is 20% of $1460, or $292. The problem is not optimally worded. This issue should be resolved before the mathematics is performed. You might not have noticed the ambiguity without careful scrutiny.

Let's take a look at how crucial scrutinizing the following problem carefully is.

> Eve is planning a rectangular flower garden that will contain a 36-square-foot flower bed. She wants to enclose the garden with a short, decorative fence that is sold in 1-foot lengths that snap together. Each 1-foot length costs $3.99, plus 8% sales tax. Find the dimensions of the garden that requires the least amount of fence.

This problem will be solved later in the chapter. Note that the solution is not the cost of the fence, although that could be inferred in a quick read of the problem. Such inferences lead to wasted time and effort and incorrect answers. Careful scrutiny helps you avoid such errors. Read the following problems. We will not solve them here; our aim is to scrutinize them. See whether you can detect a potential source for errors in each problem.

> A container in the shape of a right circular cylinder with radius 3 meters and height 11 meters is being filled with water at a constant rate of 40 cubic feet per minute. At what rate is the water level in the cylinder rising when the water level is 6 meters high?

A careful look will show that the units are different—the dimensions of the cylinder are given in meters, and the fill rate is given using cubic feet. One of these units must be converted before you begin working on the problem. Also, you may have noticed that the water level rises "at a constant rate," so the height is unnecessary to solve the problem.

> Nick has 100 feet of flexible fencing that he will set up on the lawn temporarily to form a play area for his baby sisters. What is the area of the largest play area he can enclose with 100 feet of fencing?

Students often assume that the play area must be rectangular. In fact, a circular play area with a circumference of 100 feet will contain more area than any rectangular play area with a perimeter of 100 feet.

Just as you shouldn't make assumptions that are unwarranted, also be careful not to make generalizations too quickly. Examine this problem:

> Compute each of the following:
>
> $$(20 + 25)^2$$
> $$(30 + 25)^2$$
> $$(40 + 25)^2$$

The answers to the first two expressions are 2025 and 3025, respectively. Study these two examples. Can you formulate a conjecture? Did you notice that the answer is composed of the four digits in the parentheses, in the same order? If you use this conjecture to compute $(40 + 25)^2$, you will make an error. The answer is not 4025 but rather 4225, and the answer does not follow the pattern. You shouldn't base a conjecture on too small a pool of trials. Don't be quick to generalize—scrutinize!

> Two competing pizza stores want to attract customers. One offers a free 10-inch-diameter pizza with the purchase of each regular pizza. The other has increased its pizza's diameter by 4 inches and not raised the price. Which store offers customers the better deal?

Careful scrutiny reveals that there is insufficient information to solve this problem. What is the original diameter of each store's regular pizza? What is the price per square inch of each deal, and how can it be computed without any prices being given? Sometimes advertisements are deliberately evasive so that consumers will have more difficulty making comparisons. The problem as posed has no solution.

> Two trains are riding on parallel tracks. One left Baltimore for New York City at 11 A.M., and the other left New York for Baltimore at 12 P.M. New York and Baltimore are 210 miles apart. The first train traveled at 60 mph, and the second traveled at 65 mph. At the time when the two conductors meet each other, which conductor is closer to New York?

It is astonishing to watch excellent math students work on this problem, applying all sorts of algebraic formulas, diagrams, variable representations, and so on. If you read the problem carefully and scrutinize it, you might notice that *when the conductors meet each* *other they are the same distance from New York!*

When scrutinizing, it is important not to detrimentally narrow the scope of your thoughts. Be open-minded about possible solutions. If you make an assumption, don't make it ironclad. Possibly you are thinking along a track that will hinder your quest for the solution. The following problem illustrates this situation:

Find the rule that explains the following sequence:

$$8, 5, 4, 9, 1, 7, 6, 3, 2, 0.$$

The solution will be given in this paragraph; if you want to pursue the solution on your own, stop reading now. Invariably, you will start by applying arithmetic rules to try to "coax" an algorithm that these numbers follow. Possibly there is one, although this author has never seen it. Open your mind. Think liberally. Don't think arithmetic—think spelling. The above sequence is the single-digit whole numbers arranged in alphabetical order according to their spelling! Our point is to warn you against considering only one course as you search for a solution. Expand your sights. You might need geometry to solve a probability problem, or calculus to solve an algebra problem. Always consider a broad range of options.

Although you might consider some of the examples here to be "trick" questions, they illustrate the need to scrutinize problems. Working on a question that you inferred rather than on the problem that was intended is not prudent. Solving problems can be hard enough without adding such "unforced errors." As your mathematical sophistication grows and you begin to tackle higher-level problems, careless reading can cause you to waste valuable time and effort. It is important to assess each problem by reading it carefully, critically, and fully.

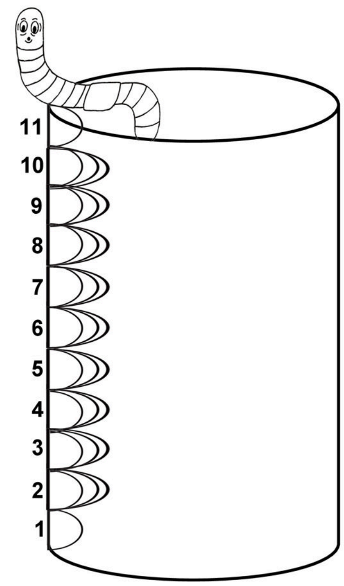
Problem-Solving Strategies

We will discuss several problem-solving strategies in this section, but this overview could not possibly provide all the experience you need in problem solving. You'll need to draw on other experiences in order to become an accomplished problem solver. As mentioned earlier, you should read some of the recommended works on problem solving. Keep in mind that familiarity with problem-solving strategies will help you handle the new, non-routine claims and conjectures you meet in your research. Included in the discussion of the strategies are some sample problems. If you would like to work through the problems before reading the solutions, stop reading after each problemis posed.

Draw a Diagram

Most students are in the habit of drawing diagrams in geometry problems, and rightfully so. A picture is truly worth a thousand words, and a diagram can help you discover a solution because the visual images enhance the verbal explanations. Diagrams can help solve non-geometrical problems as well. Let's see how drawing diagrams can help us solve the following problems.

> A worm is at the bottom of a 12-foot-deep well. Each day it crawls up 2 feet, but during the night it slips back 1 foot. In how many days will the worm get out of the well?



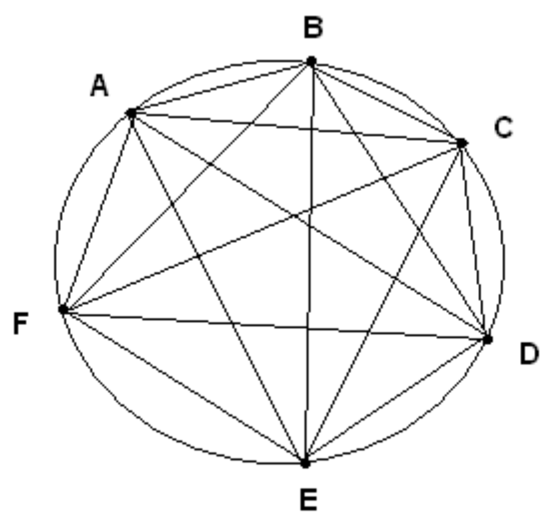
Solution: It seems that the worm gains 1 foot per day. This reasoning accounts for the popular but incorrect answer of twelve days. Let's draw a diagram and follow the worm's progress day by day. We will label 1-foot increments on the side of the well and draw on the diagram to follow the worm's progress. Each arc represents a trip the worm made between the foot markings. So on the first day he went up to 1-foot, and then to the 2-foot mark, but slipped back to 1 foot. The next day he made another trip from 1 foot to 2 feet.

As you can see, on the eleventh day, the worm is at the top, gets out, and does not slip back! The answer is therefore eleven days, not twelve.

Let's look at another problem for which drawing a diagram really helps.

> There are six officers in a school's student government organization. A committee made up of two of them is to be randomly selected to represent the student government organization at local school board meetings. How many different committees of two are possible?

Solution: We can draw a circle and place the six people, represented by six letters of the alphabet, around the circle. Next we draw a line segment from each person to each of the other students. Each line segment represents one committee. Now we simply count the number of line segments to find the number of committees.



The answer is fifteen different committees of two. The different committees can be listed by reading each line segment in the picture.

Make a Table

A table is a systematic way to list the different possible scenarios of a problem. Examination of a table can help you find the solution to a problem. Let's look at a problem posed previously in this chapter. Keep in mind that you might want to draw a diagram *and* use a table to find the solution, or use just one of the strategies.

> Eve is planning a rectangular flower garden that will contain a 36-square-foot flower bed. She wants to enclose the garden with a short, decorative fence that is sold in 1-foot lengths that snap together. Each 1-foot length costs $3.99, plus 8% sales tax. Find the dimensions of the garden that requires the least amount of fence.

Solution: As mentioned previously, there is unnecessary information that can be eliminated from this problem. The task is to find the length and width that requires the least amount of fencing, not to find the cost of the fencing. (Eliminating unnecessary information is a problem-solving strategy in itself that is discussed later in this chapter.)
Let's set up a table of possible lengths and widths and the resulting perimeters. Remember that the length and width selections are not arbitrary; they must give an area of 36 square feet and must be in 1-foot increments.

Table 3.1. Perimeters Based on Different Lengths

Length	Width	Perimeter
1	36	74
2	18	40
3	12	30
4	9	26
6	6	24

A square measuring 6 feet on each side gives the minimum amount of fencing, 24 feet. The table organized the information and made it easier to analyze.

Use a Matrix

Closely linked to the "make a table" strategy is the "use a matrix" strategy. A rectangular array with rows and columns is called a **matrix.** Like diagrams and tables, matrices can help clarify information. The entries in a matrix can be symbols, numbers, or letters, depending on the particular problem. Let's see how a matrix can help us solve the following problem.

> Janet, Rich, and Linda are students who have different hobbies and different pets. Janet's hobby is painting, and she does not have a bird. The student whose hobby is gardening does not have a dog. Linda's hobby is not gardening. The person whose hobby is skiing has a cat. Can you determine the hobby and pet for each person?

Solution: Scrutinizing this problem should tell you something. This story is too convoluted to try to decipher without a table. Underline the hobbies and pets—they will help us fill out the table. We will use initials to represent hobbies, people, and pets, the items to be compared. The strategy is to set up a matrix. There are different matrices we could use. Let's start with a simple matrix for matching pets and people:

	J	R	L
B			
D			
C			

Some clues match hobbies with pets or hobbies with people, so we'll add two more matrices for matching hobbies.

	J	R	L	G	P	S
B						
D						
C						
G						
P						
S						

Number the sentences in the problem from 1 to 5. We will fill in an X or a √ in each box for no and yes, respectively. Copy a matrix like the one above on scrap paper and fill it in as we explain the solution.

Sentence 2 tells us that under Janet's name we should place an X next to bird (B) and a √ next to painting (P). Since Janet is the painter, Rich and Linda are not, so X's can be placed next to painting for Rich and Linda. Also, Janet is not the gardener or the skier. Sentence 3 tells us to put an X on the dog row under gardening. Sentence 4 tells us to place an X next to gardening under Linda. Now we see that Rich must garden and Linda must ski, so we place √'s in those boxes. Sentence 5 tells us to enter a √ in the cat row under skiing. As a result, the painter and the gardener receive X's in the cat row.

Therefore, the gardener (Rich) must have a bird (√). Finally, the painter (Janet) has a dog.

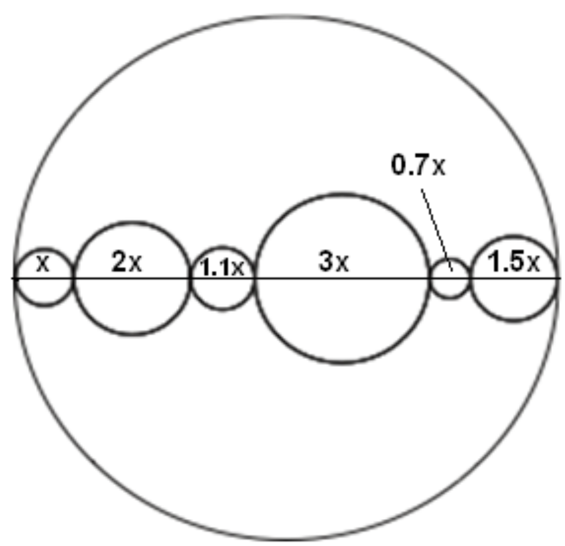
	J	R	L	G	P	S
B	X			√		
D				X	√	
C				X		√
G	X	√	X			
P	√	X	X			
S	X		√			

At this point, the rest of the chart could be filled in by correlating information from one part of the chart to another. But enough information is here already to allow us to state a solution: Janet is a painter and has a dog, Rich is a gardener and has a bird, and Linda is a skier and has a cat. How difficult would this problem have been without the matrix?

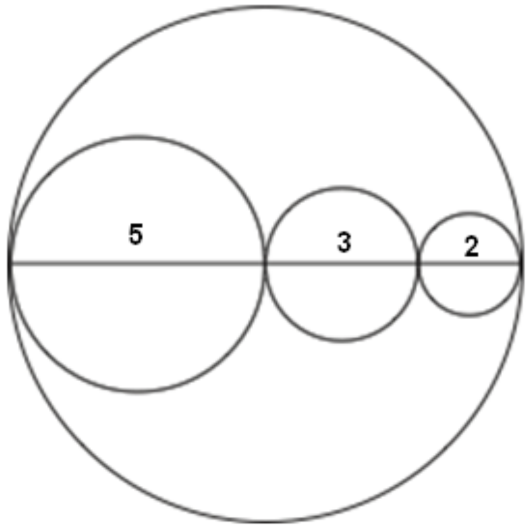
Solve a Simpler, Related Problem

Certain problems may overwhelm you due to large numbers and/or a variety of conditions. Solving a simpler, related problem affords you the chance to reason out a strategy that might extend to the original problem. There are several ways to create a simpler, related problem. You might use a number that is easier to manipulate in place of a given number in the problem. You might replace a variable with a number, work out the problem with that number, and then apply the process you used to the variable in the problem. You can also eliminate unnecessary information and/or change some of the conditions of the problem. Try using the "solve a simpler, related problem" strategy to solve the following problem.

Find the difference between the circumference of the large circle below and the sum of the circumferences of the smaller circles. The diameters of the smaller circles are given in terms of x.



Solution: To analyze this problem, let's begin by simplifying it. Suppose we reduce the number of small circles and assign numbers to the variable diameters, relaxing the ratio conditions placed on the diameters of the small circles. These three changes create a simpler, related problem. We now solve the related problem.



The diameter of the large circle is 10, so its circumference is 10π. The circumferences of the three smaller circles are 3π, 5π, and 2π. The sum of these circumferences is 10π, which equals the circumference of the larger circle. The difference is therefore zero. Is this result a coincidence based on the numbers we picked? This is always a concern when you change a problem; however, in this case, you should pick other numbers and verify that the difference is still zero. This property can be shown algebraically. Often the solution process to a numerical, related problem will get you more prepared to *attempt* an algebraic solution. You'll find this happening in the course of your math research.

Let's look at a more difficult example and its solution using the strategy of solving a simpler, related problem.

How many consecutive zeros are at the right end of the number represented by 33!?

Solution: The number 33! is too large for your calculator. (What factorial *can* your calculator express without using scientific notation?) As you scrutinize the problem, you will realize that to try to find the product by repeated multiplication would be cumbersome and prone to error based on the sheer volume of computations. Underline the words *consecutive* and *right*. The plan is to work on solving the same problem with a smaller number, say, 12!, and see if we can extend the solution to 33!. Write out 12! as a product of consecutive numbers:

$$12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

The 10 should speak to you. The zeros we are looking for are formed by factors of 10. Does this mean that 12! has only one zero at the right? Think. Scrutinize. Think about the prime factors of 10, that is, 2 and 5. If each factor of 12! were factored into its prime factorization, we could look for sets of 2 and 5, knowing that each set will represent one zero at the end of 12!. Think more critically. There are many even numbers in the prime factorization, so there should be many factors of 2. A factor of 2 can create a zero only if it can be paired with the factor 5. The factors of 5 seem much more scarce. If we count the number of times 5 appears in the prime factorization and thereare enough factors of 2 to pair them up, we can figure out the answer to the problem for 12!. The factors 10 and 5 each have one 5 in their prime factorization, so 12! has two 5s in its prime factorization. When these two 5s are paired with any of the even factors, two consecutive zeros are created on the right. The number 12! thus has two consecutive zeros on its right end.

This strategy extends without much modification to 33! Could you solve the problem with 33! without writing out the factors? Try it. See if you can verify that the solution is seven consecutive zeros.

Use Trial and Check

You may have heard of this problem solving strategy as "trial and error" or "guess and check." Hopefully your trials won't be random guesses, but solutions that sound like they are "in the ballpark" of possibilities. So we prefer to call them trials rather than guesses. Additionally, when you find out that some number *isn't* a solution, you *have* found useful information (you know that number isn't correct) so you have not made an error. Often the trials help steer you in the direction of better numbers to try. You haven't made an error but are actually a step closer to the correct answer!

If there is a relatively small pool of possible answers to a problem, the trial and check method can be very effective. It is sometimes more prudent to try the few possible answers than to construct a lengthy theoretical argument to find the solution. Contrast the trial and check strategy with that involved in solving a simpler problem. We solve simpler, related problems when some complexity in the problem may hinder our ability to find a suitable strategy. Using trial and check is more suitable when a problem lacks complexity and has a small number of possible solutions we can try. The trial and check method often involves eliminating possibilities. If you can rule out some of the trials, you should. Hopefully, there will be some reasoning behind your trials; your trials shouldn't all be mere guesses, if possible. Try to solve the following problem using trial and error.

> In ten years, Nancy will be four times as old as Bernadette is now. Bernadette's age is now two years less than half of Nancy's age. How old will Bernadette be in twenty years?

Solution: Scrutinize—remember that ages are positive integers. Also, note that if in ten years Nancy will be four times as old as Bernadette is now, then your first guesses for

Bernadette's age should be less than twenty-five. We will try different ages, check each trial, and keep track of our trials in a table. We will use ages for "Bernadette now" as trials and enter them in column 1. Then we can fill in columns 2, 3, and 4 and check to see if the conditions of the problem are satisfied. If they are, columns 1 and 4 should match. We can rate our guesses in the fifth column. Take a look at the following trial table.

Table 3.2. Age Comparisons

Bernadette Now	Nancy in 10 Years	Nancy Now	2 Less Than Half Nancy's Age	Rating
10	40	30	13	close
12	48	38	17	not as close
8	32	22	9	very close
7	28	18	7	correct

Note that our second guess was farther off than the first, prompting a third guess in the other direction. Verify that the last line of the table satisfies the conditions of the problem. Bernadette will be twenty-seven years old in twenty years. Keep in mind that making a systematic list is often part of the trial and error strategy, because you need to keep track of the trials you have completed. You don't want to repeat any trials you have already tried, and you also want to allow yourself to see any possible pattern in the results.

Look for a Pattern

When an answer to a problem cannot be found directly, it is sometimes helpful to examine a sequence of solutions to simpler problems that lead up to the solution to the original problem. This strategy uses, in part, both the "solve a simpler problem" strategy and the "make a table" strategy because it is usually helpful to put the results in a table. Let's look for a pattern to solve the following problem.

Find the sum of the first hundred odd whole numbers.

Solution: As you read the problem, you will realize that you could write out the first hundred odd numbers and find the sum; however, there is a chance of error when you are adding so many numbers, even with a calculator. Let's look at the sum of a smaller number of odd whole numbers and see if we can gain any insight into the problem. Look at the accompanying table.

Table 3.3. Analyzing Addends and Sums

Number of Addends n	Expressions for Sum	Sum
1	1	1
2	1 + 3	4
3	1 + 3 + 5	9
4	1 + 3 + 5 + 7	16
5	1 + 3 + 5 + 7 + 9	25

Do you notice a pattern? If n represents the number of odd whole numbers to be added, the sum is n^2. Try a few other examples to verify this conjecture. The table does not constitute an algebraic proof, but it does give you a reason to try to create a proof. (We discuss proofs in greater detail in Chapter 6.) The first hundred odd whole numbers have a sum of 100^2, which equals 10,000.

Use Algebra

The algebraic skills you have learned are very useful problem-solving tools. Let's use algebra to solve the age problem first posed in the trial and error section. (We've already shown a solution using trial and error that included making a table.)

> In ten years, Nancy will be four times as old as Bernadette is now. Bernadette's age is now two years less than half of Nancy's age. How old will Bernadette be in twenty years?

Solution: The reading step is crucial here, because you must first translate the words into algebraic expressions that accurately reflect the conditions of the problem.

Let b = Bernadette's present age.

Let n = Nancy's present age.

Underline the key phrases that you will use to create the equations.

Bernadette's age is now two years less than half Nancy's: $b = \frac{n}{2} - 2$

In ten years, Nancy will be four times as old as Bernadette is now. $n + 10 = 4b$

If we substitute for b in the second equation, we have a linear equation in one variable:

$$n + 10 = 4\left(\frac{n}{2} - 2\right)$$

$$n + 10 = 2n - 8$$

$$n = 18$$

By substitution, $b = 7$. In twenty years, Bernadette will be 27 years old.

Notice that you could have used the table from the trial and error section and filled it in with variables. Very often, the numerical manipulations required to fill in a table can help you determine variable representations for the problem. Let's add the variable representations to the trial and error table from the age problem previously discussed.

Table 3.4. Algebraic Analysis of Age Problem

Bernadette Now	Nancy in 10 Years	Nancy Now	2 Less Than Half Nancy's Age
10	40	30	13
12	48	38	17
8	32	22	9
7	28	18	7
b	$n + 10$	n	$\frac{n}{2} - 2$

From the first two columns, we know that 4b = $n + 10$. The first and last columns yield the equation

$$b = \frac{n}{2} - 2.$$

Notice that these are equivalent to the equations formed above and can be solved similarly.

When you complete a problem that you solved algebraically, verify that the answers fit the conditions of the problem. Why can't you simply substitute the answers into the equations to check the solution? Keep in mind that the most difficult part of using algebra is setting up the correct expressions and equations. Read and scrutinize carefully.

Work Backward

Did you ever know the answer to a problem and use it to figure out how you could do the problem? If so, then you have already worked backward to find a solution. Working backward is helpful when an end result is known and when you know the steps for getting from the unknown to the end result. Let's look at an example.

> A furniture clearance center adjusts its prices weekly according to the following markdown schedule:

Week 1 price (W_1)—price is as marked
Week 2 price (W_2)—10% off week 1 price
Week 3 price (W_3)—20% off week 2 price
Final price (F)—25% off week 3 price

Beth bought a couch that had been in the store for five weeks for $324. What was the original marked price of this couch during week 1?

Solution: Since we know the end result (the price Beth paid) and we know the rules used to arrive at it (the markdown schedule above), working backward to find the original price seems like the appropriate strategy. Since the couch was at the store for more than four weeks, let's work backward from the final price of $324.

The final price of $324 is 25% off W_3. Therefore, it is 75% of W_3.

$$0.75\ W_3 = 324$$

$$W_3 = \$432.$$

The week 3 price of $432 is 20% off W_2. Therefore, it is 80% of W_2.

$$0.80\ W_2 = 432$$

$$W_2 = \$540.$$

The week 2 price of $540 is 10% off W_1. Therefore, it is 90% of W_1.

$$0.90\ W_1 = 540$$

$$W_1 = 600.$$

The original marked price of the couch at the clearance center was $600.

You probably have noticed that some of the solutions above used more than one strategy. Many solutions will comprise combinations of the strategies discussed. How will you know which strategies to use? Reread Polya's statement that opened this chapter. It is a SUPERB commentary on attaining problem-solving proficiency.

A Look at Looking Back

The last part of solving any problem is to read the problem one more time to make sure you answered the question. This is part of reviewing—looking back and checking your work. Regardless of what problem-solving technique you employ, it is always a good idea to see if the answer you find makes sense. Let's review the age problem presented earlier in this chapter.

> In ten years, Nancy will be four times as old as Bernadette is now. Bernadette's age is now two years less than half of Nancy's age. How old will Bernadette be in twenty years?

The answer is that Bernadette will be twenty-seven. The specific method of solution is not our focus here. Before checking the answer against the conditions of the problem, we want to see if the answer makes sense. First, the age of twenty-seven for a human being makes sense. An answer of less than twenty years would not make sense. (If Bernadette is alive now, in twenty years she must be at least twenty.) We would also probably be uncomfortable with an answer of 347 years old, which could be arrived at as a result of an error. After examining this level of "Does the answer make sense?" we can check the answer against the problem to see if it satisfies all the conditions.

Sometimes it is more difficult to determine whether an answer makes sense. You may not have a frame of reference, as you did with the age problem. For example, suppose you did a report on the Empire State Building, a 102-story skyscraper in New York City. How many miles of electrical wiring are in the Empire State Building? Three? Sixty? Four hundred? Two thousand? It's difficult to have a handle on whether a particular answer makes sense in this case. With this in mind, read through the famous "Birthday Problem."

> How many randomly selected people would you have to assemble to make the probability of two or more people having the same birthday (month and day) greater than 50%?

There are 366 possible birthdays, including February 29. Scrutinize. The answer can't be greater than 367 people, because we are guaranteed a match with 367 people. How many people do you think it would take to get the probability over 50%, where there is a better chance of a match than no match? More than 180? Fewer? The answer is 23. The probability of at least two people having the same birthday is greater than 50% if there are just twenty-three random people polled. Try this in your classes that have more than twenty-three students. You'll be surprised at how few people it takes to create this matching birthday situation! The mathematical solution requires a background in counting principles of probability. If you have learned about permutations, combinations, and the probability of a complement, you can attempt the solution. It is also easy to find on the Internet.

We discuss this here because the correct solution seems unreasonable. Most people think 23 is too small a number, that it's incorrect. The theoretical mathematics makes the solution clear, and if you experiment throughout your classes (many classes), you will conclude it is correct. This is an example of a situation when "looking back" requires you to examine your strategies and calculations for correctness, because the answer may not seem to make sense.

The following problem is a classic with a counterintuitive answer.

You are offered a job for the month of July—thirty-one days of work. You will receive your pay at the end of each day. You can choose between two payment options:

> Receive $1,000 per day for each of the thirty-one days, *or* receive $.01 the first day, $.02 the second day, $.04 the third day, $.08 the fourth day, and so on. Each day's pay will be double the pay of the previous day. In either case, you are being paid each day.

Which payment option will earn you the most money by the end of the month?

Calculate the total pay for the first option. For the second option, draw a thirty-one-day calendar and use your calculator to determine each day's pay. Enter each day's pay on the calendar. You will probably be able to tell which option is better without adding up all of the payments for the second option. Is the answer surprising?

A related problem involves the thickness of a sheet of paper. First, you will need a problem-solving strategy that can be used to determine the thickness of a sheet of paper. Also, remember that there are 12 inches in a foot and 5280 feet in a mile.

> The earth is approximately 93 million miles from the sun. How many times would you have to fold a sheet of paper so that the total thickness is greater than 93 million miles?

If you try to physically fold the paper, you will find that it is a cumbersome chore after just a few folds, so the rest of this problem needs to be done conceptually. How amazing is this answer? Compare this problem to the "penny doubled every day" problem and look for similarities!

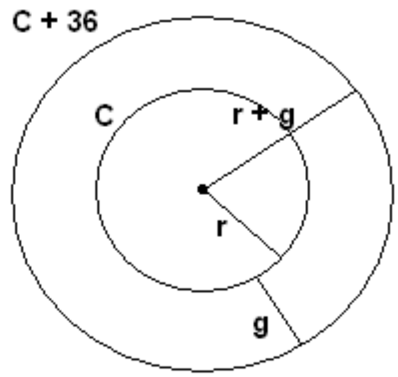
Try this popular circumference problem, which also involves the earth.

> A steel band is placed around the earth, snugly fit at the equator. (The equator is approximately 25,000 miles in circumference.) The band is cut, and a 36-inch piece of string is spliced into the steel band. This new circular band is placed around the earth, centered off the earth's surface, so its center coincides with the center of the earth. A gap is created between the equator and this circular band. What could you fit in this gap? A hair? An index card? How wide is this gap?

The gap is approximately 6 inches wide! In fact, the gap is the same width as the gap created if 36 inches of string are added to a band placed around a *nickel!* How could 36 inches have such a profound effect over 25,000 *miles?* How could they have the same effect on the relatively tiny nickel's circumference? You can take some string and do the experiment around a trash-can cover and a nickel to verify the result. Mathematically, the solution is not complicated.

$$C = 2\pi r$$

If the circumference increases 36 inches, the radius must increase. Let's call the increase in the radius *g,* for gap.



The new, larger radius is therefore $r + g$. We can create an equation for the larger circle, based on the circumference formula..

$$C + 36 = 2\pi(r + g)$$

$$C + 36 = 2\pi r + 2\pi g$$

Since $C = 2\pi r$, we can subtract these equal quantities from both sides of the equation.

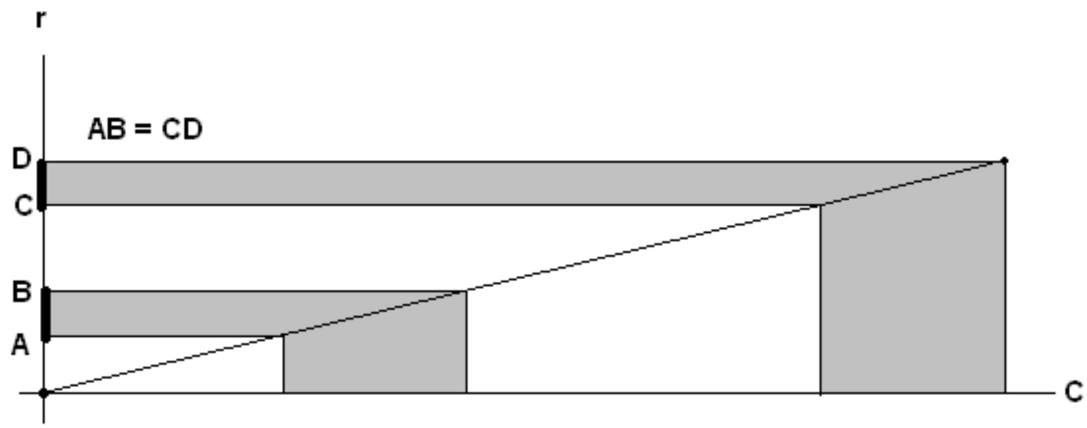
$$\begin{array}{rcl} C + 36 &=& 2\pi r + 2\pi g \\ \underline{-C \quad} &\underline{=}& \underline{-2\pi r} \\ 36 &=& 2\pi g \end{array}$$

Divide both sides by 2π to solve for g. Since 2π is approximately 6, the gap is just under 6 inches! The formula shows why the original circumference does not affect the answer; hence, the nickel and the earth yield the same gaps.

Drawing a graph also shows why this answer makes sense. Since $C = 2\pi r$, we can write the following:

$$r = \frac{1}{2\pi} C$$

The graph of this equation on the *r, C* axes is a straight line with slope 1/(2π) and *y*-intercept zero. The graph shows that any 36-inch increase in circumference causes the same (approximately 6-inch) increase in the radius, regardless of the original circumference.



The gap is, surprisingly, 6 inches, and this result totally defies your mathematical intuition!

Try this problem on your friends and family. The answer is amazing, so don't be surprised if people tell you your solution is wrong! Again we see that correct answers sometimes aren't what we expected, and we may think they don't make sense. In such cases, we must follow Polya's steps and check all the strategies and calculations. The following problem is another classic with an unexpected result.

Three people pay $10 each (total $30) to check into a hotel. The manager realizes that they were overcharged and gives $5 to the bellboy to return to the three people. The bellboy decides to return $1 to each of the three people and keeps $2 for himself. The three people paid $9 each, for a total of $27. The bellboy received $2. What happened to the other dollar?

We leave this problem for you to scrutinize and solve.

Problems and Questions

Did you ever wonder why we *park* on a *driveway* and *drive* on a *parkway?* Why is it that when you send something by *car* it's a *shipment* and when you send something by *ship* it's called *cargo?*

Think about what you've read in this book. Chapter 1 focused on questions, and Chapter 2 focused on strategies to find answers. Notice how many questions were asked and how many were inferred. Problems inspire questions, and these questions can become new problems. In carrying out thorough mathematics research, you will face new questions that arise from your work. As mentioned in the "build" step, you should write down these questions. You may decide to answer them as part of your research. Practice writing questions. See how many questions you can create based on any of the problems in Chapters 1 and 2. Asking questions is a sign that you are thinking open-mindedly—you are entertaining other scenarios. Good, pertinent questions are an indicator that you understand the problem. There is an art to formulating a clear question that has well-defined conditions. Practice this as you problem-solve and as you do your research. For further insight into asking questions, read *The Art of Problem Posing,* by Stephen Brown and Marion Walter.

Try writing your own problems. You can use an original extension of a given problem to create a new problem or think of some problems either on your own or with your classmates. Ask your math instructor, department chairperson, or math team coach to act as an editor and reviewer for your problems. You might want to have your math class start a problem-solving column in your school's newspaper, using your original problems. Maybe you can solicit a local business to sponsor a monthly prize for the winner of a newspaper problem-solving contest. Your class can publish its own problem solving booklet, using problems created by the students, to be distributed to students, instructors, and administrators.

The National Council of Teachers of Mathematics (NCTM) publishes a journal called *Mathematics Teacher*. Each month, a calendar features a different original problem for each day of that month. Try sending some of your class's original problems, with solutions, to the NCTM for possible publication in the calendar. Go to www.nctm.org for more information. It is truly exciting to see your name and your problem in print—a problem that will be read by people all over the United States and Canada!

Be Determined

Musicians practice. Athletes practice. Mathematicians practice. Sometimesa difficult challenge leads to frustration. Perhaps you found some of the problems in this chapter frustrating. Good researchers and problem solvers build confidence and perseverance when their previous problem-solving persistence leads them to correct solutions. You'll need much more problem- solving practice than this chapter can provide. You should seek out books whose primary focus is problem solving. Problem-solving and brainteaser books are readily available in bookstores. Learning how to play mathematical games will sharpen your insight and build your concentration and persistence levels. Some excellent games are available at bookstores and hobby stores. React constructively to the challenges you face; the teasing and frustration of a seemingly unsolvable problem should trigger determination. Ask questions. Form groups of students that can discuss the problem and exchange or pool ideas.

Don't set a time limit on finding a solution—it's not realistic. Sometimes you will have to leave a problem and come back to it later; therein lies the reason for keeping legible notes. Frequently, when you leave a problem, an idea for a solution will come to you at a time when you are engaged in some other activity, because all the while the problem was on your mind. You may give up on a problem, *or think you have given up,* only to have a solution come to you when you are playing softball, listening to music, or walking to school. At that point, you've reached the epitome of the spirit of problem solving; you'll be able to apply your problem-solving experience to the problems you encounter in the course of doing mathematics research.

> *A great discovery solves a great problem, but there is a grain of discovery in the solution of any problem. Your problem may be modest, but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.*
>
> —George Polya, *Mathematical Discovery,* 1961

Chapter 4
The Importance of Clear Communication

Think of the major inventions of the past few decades. Calculators. Computers. E-mail. EZ pass. Text messages. Voice mail. Tweets. Cell phones. Tablets. GPS's. Do you realize how many of these inventions involve some aspect of written communication? Why are we so obsessed with communicating? How does communication impact your math research?

Your math research will involve patterns, conjectures, proofs, errant trials, new findings, intricate diagrams, etc., and it is essential that you be able to communicate what you did effectively and empathetically. You will become so immersed into what you do that to explain it to someone with no background in your research can become a challenge. However, there are systematic strategies you can employ to learn how to convey your findings in a sequential, logical format that is totally comprehensible to someone who is unfamiliar with your research.

The best way to practice this is to apply the writing strategies to the mathematics you already know—the mathematics you learn in your core mathematics class.

Why Focus on Writing in Mathematics?

You might never have thought of the role writing should play in your mathematics classes. Writing is a component of every facet of our world. Everyone needs to know how to write. All professions require communication; none of us operate in an isolated vacuum.

You might feel that mathematics and writing are an "odd couple." How do mathematicians communicate their findings? The same way your doctor does, your teacher does, your plumber does, your babysitter does, etc. They talk. They write. They must be able to communicate their thoughts clearly.

Were you ever absent from class and asked a classmate what you missed and they replied, "I know what we did but I can't explain it."? Imagine if your car was fixed and you asked the mechanic what was wrong and she responded, "I fixed it but I can't explain what was wrong.”? How would you feel if you went to the doctor's office and your doctor said, “I know what’s wrong with you but I can’t explain it.”? How much confidence do you have in these responses?

Mathematicians must be able to communicate their findings so the results can be used to solve future problems. As you read through these sections and lessons and start your mathematics writing journey, you will be exposed to pages written by high school students just like you. You will read their work and see how well they have learned the art and science of communicating technical material. With effort and practice, you will become a pro at it, too!

Communication — An Art and a Science

Julianne Marra is 17. She is a high school junior. She is having an argument with her mother because she wants to take the car to the mall to meet her friends. Her mother wants her to work on her American History project.

“Ma, can I have the car keys to go to the mall to meet Stephanie and Nicole?”

“No, I want you to work on your history project,” said Mrs. Marra.

“But you’re so unfair. Everyone is at the mall today.” replies Julianne.

“I don’t want to hear another word about it. You’re not going to take the car!” screamed an angry Mrs. Marra.

Julianne went up to her room. She closed the door and started her history project. She was making good progress. She came upon a book listed in the bibliography of an article she was reading about Abraham Lincoln. Julianne decided that the book would be a good source for her report. She called the library to see if they had the book. They did and they put it aside for her. Julianne was going to drive over and pick it up.

“Ma, can I have the car keys?” inquired Julianne.

“I told you not to bring up the car again! That’s it. No car for a month!” her mother fumed.

Miscommunication. Families squabble because of it. Rock bands split up because of it. Countries fight wars because of it. Words have tremendous power. When used correctly, that power can be extremely beneficial. When used incorrectly, problems often arise. Has poor communication ever affected you?

- Did you ever try to program a new TV remote control based on the instructions

included with it?
- Did you ever get directions to someone’s house, follow them perfectly, and still get lost?
- Did you ever have trouble using some of the features on your graphing calculator, after reading the manual?
- Did a GPS navigation system every give you incorrect directions?
- Did you ever have a teacher who you felt had excellent command of the subject matter, but couldn’t get their message across?
- Did you ever have trouble using the information presented on the Help menu of computer software?

In these situations, and other situations like them, the person giving the instructions is so familiar with the instructions they invariably leave out some crucial information that is second nature to them. This information happens to be vital to a novice. The end result is frustration and a classic example of how expressing yourself clearly is an art, and more difficult than people casually give it credit for. Yes, you can improve the way you communicate because communication is a science too.

Had Julianne realized the science of communication, the outcome of her episode could have been altered. Had Julianne prefaced her second request for the car with an explanation of why she needed it, things could have been very different. Her mother would have realized that she was working hard on her history paper and that she was only asking for the car to go to the library; two things her mother would have been happy to hear. Instead, poor communication sabotaged everything.

Why Does Communication Sometimes Fail?

If you are a baseball fan, imagine trying to explain the rules of the game of baseball to someone who has never seen baseball being played. You’d be amazed at all of the details you would leave out, because they are second nature to you. It wouldn’t be enough to be an expert on the rules to write about them. You’d have to have an ability to empathize with someone who has no knowledge of the game before proceeding to explain the rules.

Did you ever get written directions to someone’s house or spoken directions from a GPS, follow them, and still end up in the wrong place? Often some nuance that was considered trivial was the reason you got lost.

Do you play a musical instrument? Pick a piece that you know well. Sit down at your instrument and play the piece. Then, while not near your instrument, try to write down the notes of what you just played. Often, musicians have the motions so ingrained in their brains and in their hands, they can't even tell you what notes they play; the piece has become second–nature to them!

Additionally, remember, that when you write, you don't have the benefit of orally conveying your intentions. There is no voice to listen to, no sound, and no inflection to follow. Have you ever sent, or read, an e-mail or text message that was misunderstood? The precise use of words is essential. Read the next passage to see how the lack of written precision can totally alter the intent of the writer. The story is true.

> In December 2000, the poor wording of a proposal caused voters in Suffolk County, Long Island, NY, to actually vote against their wishes. Let's explain. Suffolk County borrows money to buy open land, so it will not be used for construction and development. The county is trying to preserve natural, open spaces. The proposal was intended to "allow a limited amount of money" to be borrowed by the county to buy land for this purpose. Voters read this as an attempt to *limit* the amount the county could borrow, and voted it down because they felt the 'limit' aspect was an attempt to lessen the borrowing. When they read the proposal, the crux of the proposal seemed to be to *limit*.

The proposal was actually an attempt to allow the county to borrow money to buy open land. (They couldn't allow an *unlimited* amount of borrowing!) By voting it down because they thought a limit was an attempt to cutback the amount allowed, the voters actually voted down the approval to allow the county to borrow at all. As a result, until the next vote, the county could not borrow any money for these purposes. County legislators were quoted as saying that the resolution could have been worded better.

Usually poor communication is not deliberate. But it can be. The following example must contain an error, since it is a "proof" that 2 = 1.

Let $a = b$, where a and b are each not equal to 0.

$$a = b$$

Multiply both sides by a.

$$a^2 = ab$$

Subtract b^2 from both sides.

$$a^2 - b^2 = ab - b^2$$

Factor.

$$(a - b)(a + b) = b(a - b)$$

Divide both sides by $(a - b)$.

$$(a + b) = b$$

Substitute b for a.

$$b + b = b$$

Combine like terms.

$$2b = b$$

Divide both sides by b.

$$2 = 1$$

The algebra looks like it makes so much sense! The error was the line with division by $(a - b)$. Since $a = b$, $a - b = 0$ and you cannot divide by 0.

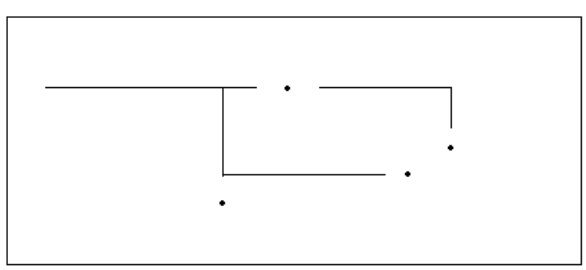
The previous little anecdotes were included just to sensitize you the need for precision in technical writing. Because mathematics is so precise, it is an excellent way to practice better writing, speaking and communicating. In Chapters 3, 4, 5 and 11, you are going to learn how to write technical information — in particular mathematics — in an efficient, productive manner. You will learn how to present the mathematics you study in a professional format that you will be very proud of. The skills you learn will extrapolate to all other disciplines, careers, and college and high school courses.

As you learn to add more words to your mathematics, you will also be amazed to find out what so many authors and teachers have found out first hand-- *when you explain material to others, you actually end up understanding it better yourself!* It's a win–win situation, as you and the person you are explaining something to both come out ahead.

A Picture Is 1000 Carefully Selected Words

To get you primed for your indoctrination to writing mathematics, we are going to try a popular communication activity. You will need a partner for this activity. Each participant must have an 8.5 x 11 in. sheet of paper and a ruler that measures in centimeters and millimeters. First, you and your partner should, together, make a list of as many geometrical terms as you can. These are any words that can help someone describe a geometrical picture. Here are some words to start off your list:

Vertical, horizontal, parallel, perpendicular, circle, triangle, square, pentagon, right angle, center, millimeters, centimeters, bisect, tangent, middle, up, down, left, right, vertex, diagonal, etc. You should get your list to include 50 to 100 words before starting the activity. In the activity, you are going to create a picture and then, sitting back to back to your partner, describe the picture you drew, one part at a time, to your partner. Here is an example:



Your partner cannot see the picture, and cannot ask any questions. You must choose your instructions carefully, and you must listen to yourself and monitor what you are saying. The only word your partner can say is "next," which indicates that you can give the next set of descriptions. Make sure that when you describe the picture, you are having your partner draw it on the same location on the piece of paper.

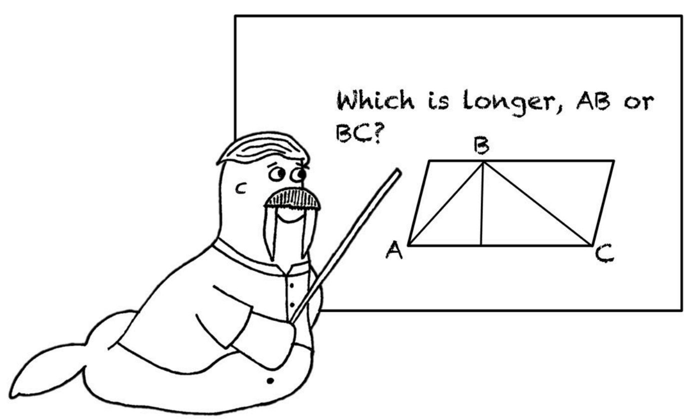
When the picture is done, you can hold both papers up to the light and see how well the pictures coincide. If your pictures don't coincide, describe what instructions were not precise. You can your cell phone to record your dialog to make sure you accurately remember the instructions. After you have compared the two drawings, switch roles and try the activity again.

It may be more difficult than you think to perfectly describe even a relatively simple picture!

If a Picture Is 1000 Words, a Paper Is 1000 Explanations

You have been writing since kindergarten. You may think, "I know how to write already." You do, but essays, short stories, poems, reports and creative stories all have specific idiosyncrasies and characteristics. Technical writing does too, and you may not have had as much practice, if any, doing technical writing. In technical writing, clarity is paramount. Your mission is to have the reader thoroughly understand what you've written. Since communication is a science, we are going to systematically approach how we improve it.

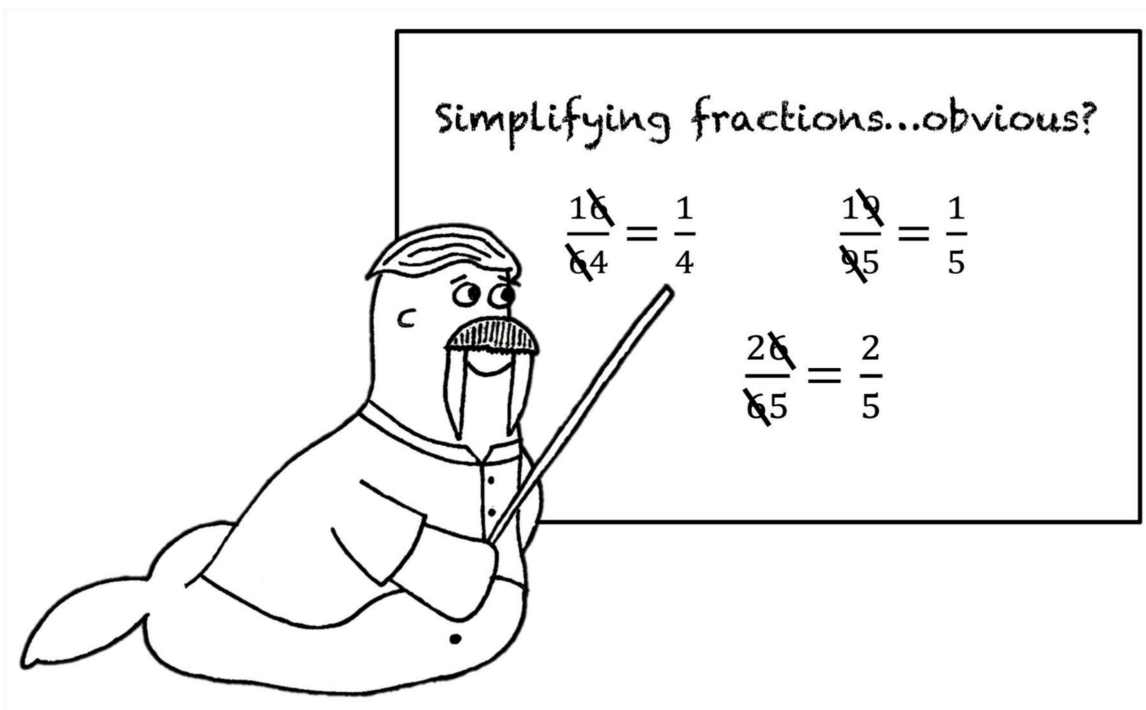
We are going to introduce some specific technical writing suggestions. All of the writing suggestions will be thoroughly explained as they are discussed. Technical writing has its own nuances. Keep in mind that the best way to improve your writing is to read some examples of good writing, and to try writing yourself.



Perhaps you remember your early elementary school years. Children who were read to the most and did the most reading optimized their ability to read with inflection and expression, and to write with expression. This is because they were barraged with examples of good language skills on such a regular basis that many good habits became second nature to them.

So take advantage of the writing tips and samples of actual student work in *Writing Math Research Papers*. You will be reading about some writing projects you can do yourself. Try doing some of these projects with the notes you take in class. As you gain confidence and develop your technical writing, imagine how you could actually improve, keeping in mind the style and flavor of the writing instructions you have mastered from working with this book. Being able to clearly explain something that you know, while being able to empathize with someone who does not know, is a specialized skill that can be learned and practiced.

"'Obvious' is the most dangerous word in mathematics."
— Eric Temple Bell (1883 – 1960)

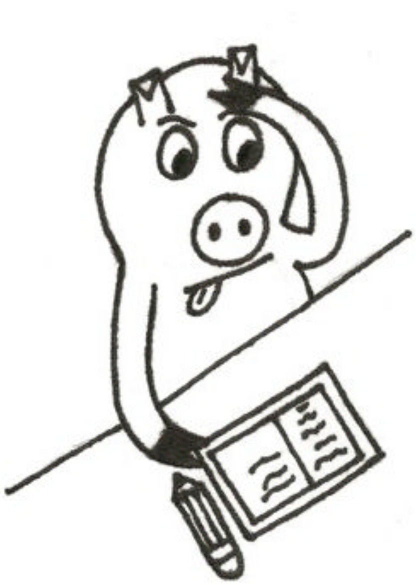


Enjoy your mathematical writing journey!

Chapter 5
The Notes You Take in Math Class

An Experiment

Your research will involve a great deal of note taking. Why is it so important to write down your findings? Do we need to take notes in class? Talk to your teacher about your interest in writing precise mathematics, and have your teacher help you with the following experiment:



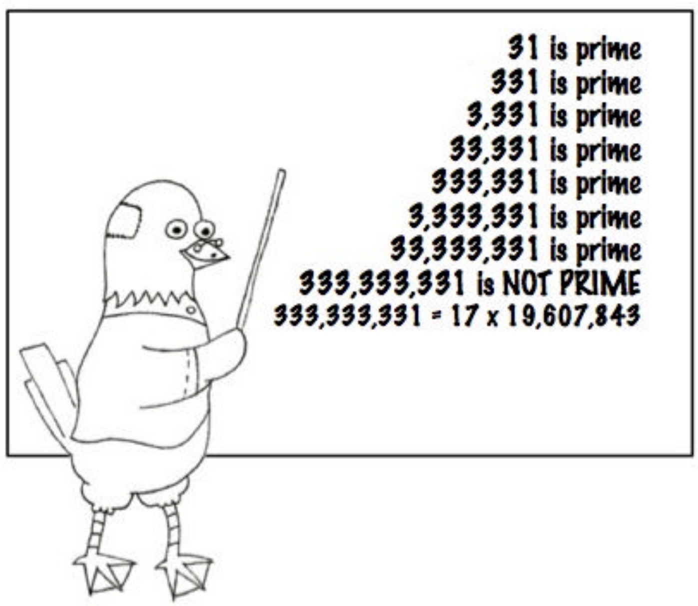
Have your teacher pick a day when a new topic will be taught. She should select several students to miss the class deliberately and have them stay in another supervised part of the school. Record the entire class lesson on a cell phone or other electronic device. The students who missed the class will listen to the recording. They will have to do the homework, based on listening to the recording, but seeing nothing about the lesson. As a class, compare the homework results of the students who were in class to the students who only used the tape. Listen to this recorded version of the lesson. See how hard it is to determine exactly what was going on because your teacher and classmates are pointing to things on the chalkboard, and using words and phrases like "put that over there" and "move this term over," etc.

Without the chalkboard in sight, the audio track is virtually useless as an instructional device. A recording of a mathematics class is generally not as useful as written notes.

Undoubtedly the only time you "write" mathematics is when you take class notes, do your math homework, or take a test. When you improve your writing techniques, it will show in all of these instances. We are going to take a look at the notes that Michael, a ninth grader, took in class. We will see how some versions are more comprehensive and understandable than others. As you read through these versions, imagine how Michael's understanding of the content improved as he created full sentences, diagrams, and other features to explain the lesson.

Michael is in an algebra class that just finished studying the slope of a line. The specific class lesson we will focus on is likely one that you have never covered, so you will really be able to see the difference good note taking can make. Eventually, you will see a version of Michael's notes that will allow you to fully understand the lesson. The lesson will not be explained here; it is up to you to see what transpired from the different types of notes Michael made.

Michael will be **annotating** his notes—he will be adding important comments, hints, diagrams, explanations, and pointing out critical concepts, and common pitfalls and misconceptions. In college you will spend many hours annotating lectures and board work presented by your professors.



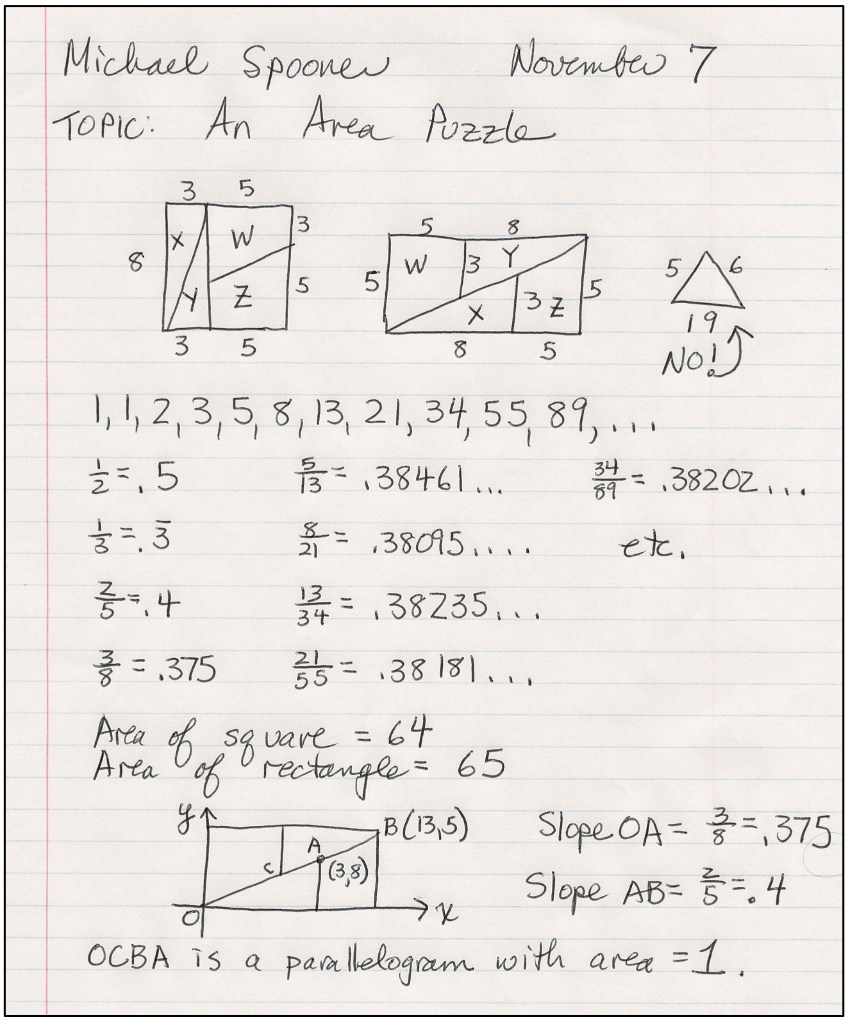
We will look at five different versions of the same class session. Each treatment has a different degree of annotations — words, sentences, phrases and diagrams that were added to what the teacher wrote on the board. These five types are:

- No Annotations
- In–Class Annotations
- At–Home Annotations
- Balloon-Help Annotations
- The Math Author Project

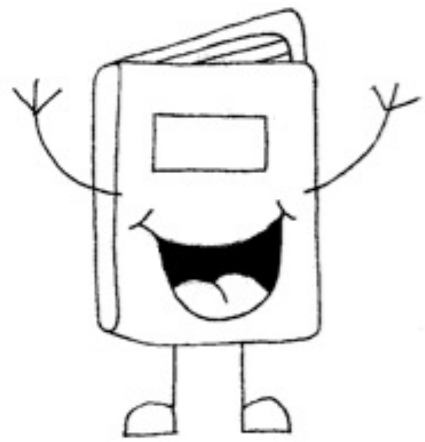
These will be explained along with sample pages from Michael's notebook.

No Annotations

Michael just copied whatever the teacher wrote on the chalkboard. This is what many high school students do. It is not the best way to take notes. Look at the notes and see how much you can decipher about the lesson. Imagine if this lesson was given in October and Michael used these notes to study for a final exam the following June. As you read the notes, think about how well they communicate. If you were absent, and copied these notes from a classmate, would you need extra help to understand what transpired during the lesson?



Don’t be surprised if you really don’t understand the lesson from those notes—you probably shouldn’t! They are a vague, disconnected conglomeration of diagrams and lists and equations. They need some explanatory sentences to become reader-friendly.



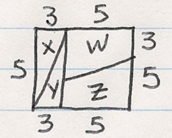
In-Class Annotations

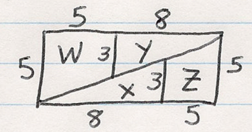
Michael copied what was written on the chalkboard, and, *during the class*, added some words that the teacher and students *said*, but did not write down on the board. These annotations will help Michael recall the lesson whenever he needs to. Look at these notes and compare them to the No Annotations version. The extra sentences Michael wrote down during the class were said during the class; he did not have to create them himself. It is a good idea to ask teachers and fellow students to repeat comments you are trying to process and write down—it shows you are focused, and it helps you formulate better sentences.

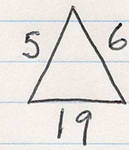
Very often, when teachers use PowerPoint to present concepts, it is difficult to copy down the material as quickly as it is presented. Consequently, you may find yourself copying material while the teacher is talking about something different. Unfortunately, since the teacher didn’t have to spend time writing a sentence (they just had to click it on PowerPoint), it is possible for them to start making other comments while you are still copying the sentence just clicked on. It is always smart to add words and phrases of your own to help clarify what the teacher said or wrote. Don’t be shy in class—if you need something restated or rephrased, raise your hand and ask! When you get to college, you’ll adopt this form of note taking and use it on a regular basis.

Michael Spooner November 7

TOPIC: An Area Puzzle







This can't exist due to Δ inequality. $5+6<19$

Area = 64 area = 65

First 11 Fibonacci #'s: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ..

F_1 F_2 F_3 etc. (F_i is i^{th} Fibonacci #)

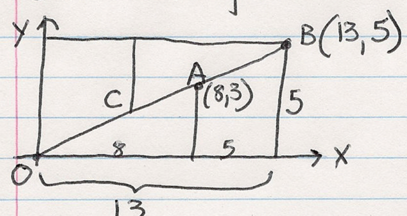
Table of Ratios in $\frac{F_i}{F_{i+2}}$ form:

$\frac{1}{2} = .5$ $\frac{3}{8} = .375$ $\frac{13}{34} = .38235...$

$\frac{1}{3} = .\overline{3}$ $\frac{5}{13} = .38461...$ $\frac{21}{55} = .38181...$

$\frac{2}{5} = .4$ $\frac{8}{21} = .38095...$

The areas of the two figures should be the same since they are made from the same 4 polygons.

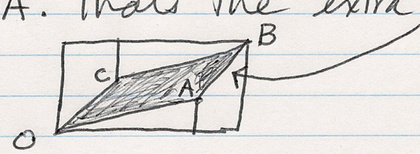


Slope OA = $\frac{3}{8} = .375$

Slope AB = $\frac{2}{5} = .4$

The slopes are different, so OB is not a straight line!

There is a parallelogram-shaped gap in the middle, OCBA. That's the extra square unit!



At-Home Annotations

Michael went home and recopied his notes, adding full-sentence descriptions to certain parts of the notes. He also referred to a diagram that the teacher had handed out on a separate sheet and discussed it in context, in his notes. He added several full sentences that better explained the "shorthand" of the original notes. If he could not express a thought in a full sentence, that was his cue to ask his teacher a question the next day. His ability to create sentences that express the math concepts acted as a barometer as to what he understood. It was very simple-if he couldn't write a complete sentence, he was unsure of the concept. Look at the full sentences and compare this version to the original notes taken in class. Not only are these notes more valuable for studying for tests, midterms, and final exams, but creating them was a form of studying in and of itself!

He did his At-Home Annotation after school on the same day of the lesson, so his memory would be fresh. After "combing" through his rewritten notes, he lightly penciled-in a question mark next to anything he didn't fully understand. The next day, he asked his teacher about these points. When Michael studies for a test, few new questions will arise.

Think about how comprehensive a system this is. It doesn't take much time, and all questions surface immediately. It's a time-efficient, effective, mature way to handle mathematics schoolwork on a daily basis. You stay current, and have an excellent set of notes you can use to recall material later on. You get your weaknesses addressed the next day.

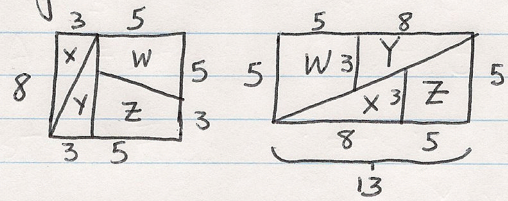
Mathematics is sequential--the new concepts are rooted in past concepts that were previously learned. When these previous concepts are not mastered, subsequent concepts are built on "quicksand." Those newer concepts are more difficult to understand because the prerequisite skills are not met.

Keep in mind that you can add homework problems, the answers, and questions and comments made about them. Examine Michael's At-Home Annotations, and compare it to the In-Class Annotations and the No Annotations versions. Notice it is twice the length of the In-Class Annotations version. Compare this version to the original notes.

Michael Spooner November 7

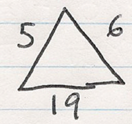
TOPIC: Using Slope to Solve an Area Puzzle

In these figures, X and Y are congruent right triangles and W and Z are congruent trapezoids.



The square and the rectangle are made up of the same 4 polygons, so their areas should be equal. But the square's area = 64 and the rectangle's area = 65.

It is possible that the picture is drawn incorrectly.



We can "draw" impossible figures by labelling them incorrectly. This "triangle" violates the Δ inequality since $5+6<19$.

In class we tried this problem to scale using graph paper. We cut out X, Y, Z, W from the square, and rearranged them to form the rectangle, and it looked fine.

The side lengths are all Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... F_i is the i^{th} Fibonacci Number.

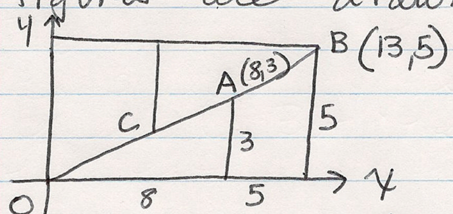
The fractions $\frac{F_i}{F_{i+2}}$, expressed in decimal form here, will help us solve the problem:

$\frac{1}{2} = .5$ $\quad \frac{3}{8} = .375$ $\quad \frac{13}{34} = .38235\ldots$

$\frac{1}{3} = .\overline{3}$ $\quad \frac{5}{13} = .38461\ldots$ $\quad \frac{21}{55} = .38181\ldots$

$\frac{2}{5} = .4$ $\quad \frac{8}{21} = .38095\ldots$

These ratios seem to approach .382... as the Fibonacci numbers used increase. These fractions can represent slopes if the figures are drawn on the x, y axes.



The side lengths are used to find the coordinates of A and B.

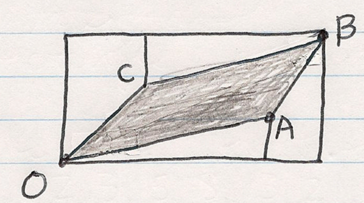
$$\text{Slope of } OA = \frac{\Delta y}{\Delta x} = \frac{3-0}{8-0} = \frac{3}{8} = .375$$

$$\text{Slope of } AB = \frac{\Delta y}{\Delta x} = \frac{5-3}{13-8} = \frac{2}{5} = .4$$

These are very close, so OB looks straight.

OAB is not a straight line – it is not the diagonal. It is made up of two "kinked" segments, but the kink is hard to see since the slopes barely differ.

This can be shown in exaggerated fashion:



The shaded parallelogram OCBA has area = 1, and this thin, "hidden" parallelogram increased the area from 64 to 65.

Notice how the three previous versions of note annotations did not involve the use of a word processor. The next two examples of annotations, Balloon-Help Annotations and the Math Author Project, will involve word processing.

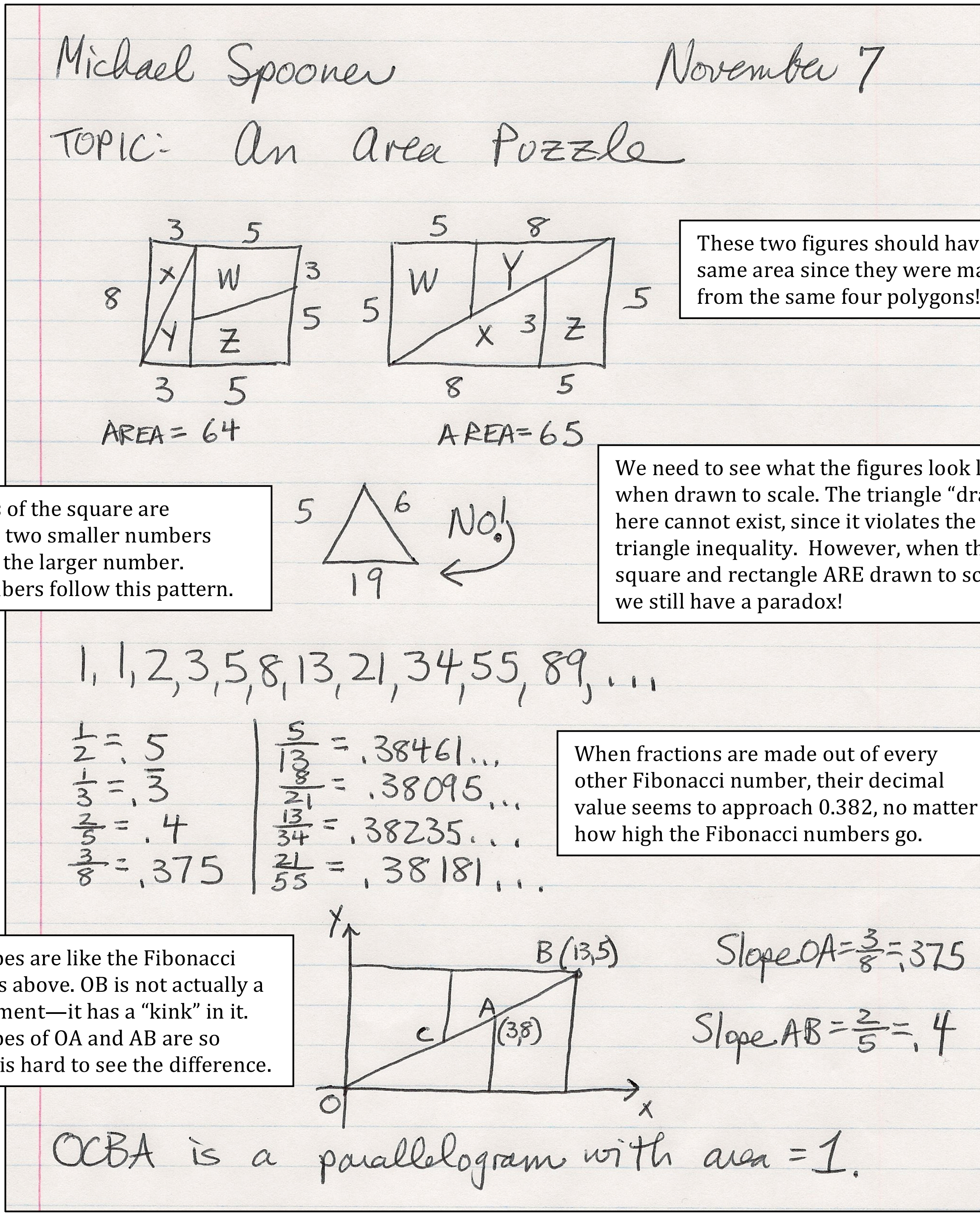
Balloon Help Annotations

The Balloon-Help Annotations add word-processed balloons to any version of the handwritten annotations. The balloons provide explanations, warnings about common pitfalls, comments, comparisons and other forms of help. When you are doing a Balloon-Help Annotations project, you can recopy your notes neatly, plan where you are going to paste up comments, and leave space for the balloons. You can word-process the balloon comments, cut them out, and paste them onto the handwritten notes. If you scan your handwritten notes, you can electronically paste up the balloons as text boxes, and print the document when all the balloons are in place.

Some students like to do Balloon-Help Annotations in a large format on oak tag. This involves a great deal of physical (not electronic) cutting and pasting. You can prepare the written work by hand or using a computer. Your teacher might want to fill a bulletin board with your Balloon Help Annotations if all students do them in this larger format. Be careful to use color with consistency and discretion. It is tempting to use colors and colored paper arbitrarily when you know your project will appear on a bulleting board. Using aesthetics to convey a point is a great idea, but it takes restraint, knowledge, and creativity!

The following example uses Michael's No-Annotations version and adds balloon comments to it. Notice how he recopied and spread out the notes, leaving space for his balloons, but all the added comments are word-processed and placed near the material they refer to. You can use arrows if your balloons need to point to something specific, or use highlighters or computer highlighting to delineate some specific aspect of the notes you want to showcase.

You could add balloons to In-Class Annotations or At-Home Annotations, but once you were doing those types of annotations, you'd probably not need balloons—you'd add any comments that would have made up your balloons right into the annotated text.


Michael Spooner November 7

TOPIC: An Area Puzzle

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

$\frac{1}{2} = .5$	$\frac{5}{13} = .38461\ldots$
$\frac{1}{3} = .\overline{3}$	$\frac{8}{21} = .38095\ldots$
$\frac{2}{5} = .4$	$\frac{13}{34} = .38235\ldots$
$\frac{3}{8} = .375$	$\frac{21}{55} = .38181\ldots$

Slope OA = $\frac{3}{8} = .375$

Slope AB = $\frac{2}{5} = .4$

OCBA is a parallelogram with area = 1.

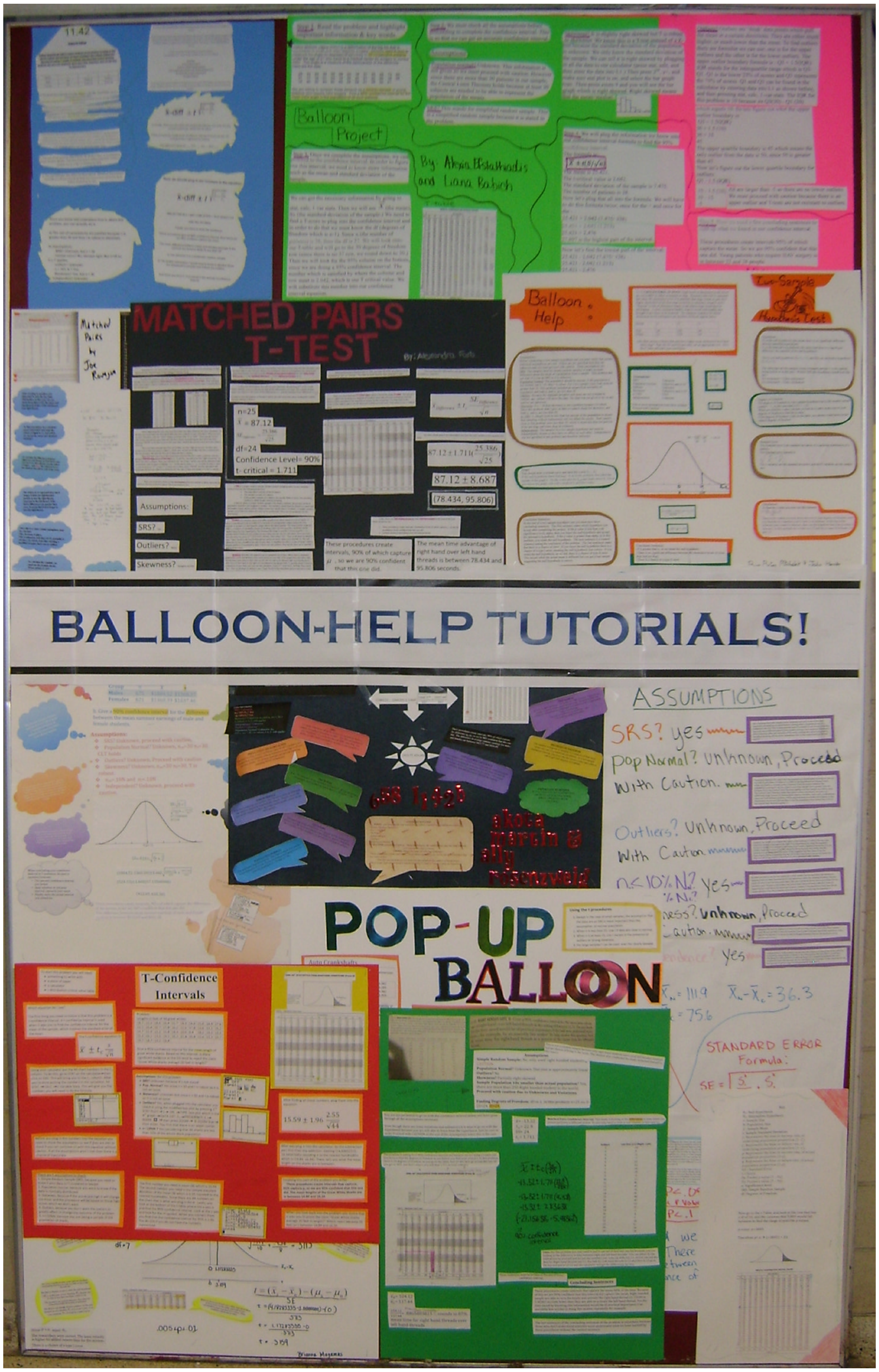
These two figures should have the same area since they were made from the same four polygons!

Since the sides of the square are congruent, the two smaller numbers need to add to the larger number. Fibonacci numbers follow this pattern.

We need to see what the figures look like when drawn to scale. The triangle "drawn" here cannot exist, since it violates the triangle inequality. However, when the square and rectangle ARE drawn to scale, we still have a paradox!

When fractions are made out of every other Fibonacci number, their decimal value seems to approach 0.382, no matter how high the Fibonacci numbers go.

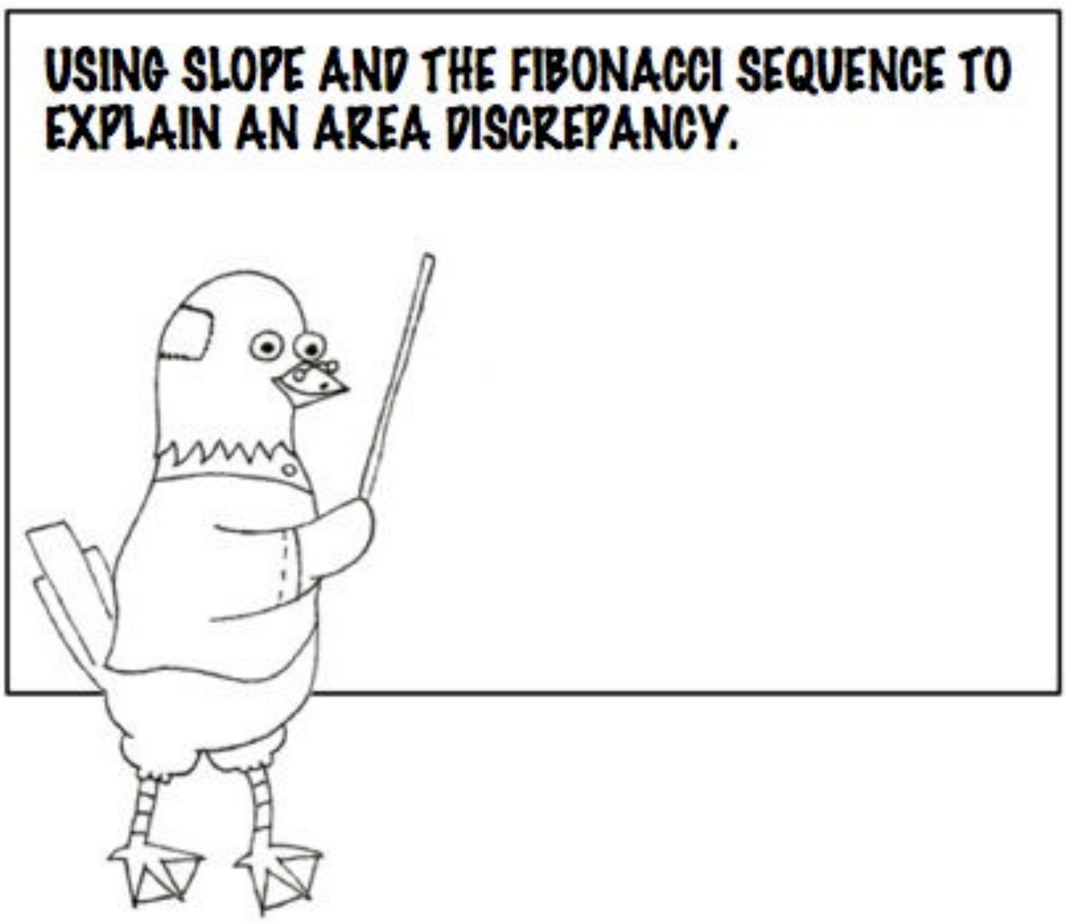
The slopes are like the Fibonacci fractions above. OB is not actually a line segment—it has a "kink" in it. The slopes of OA and AB are so close, it is hard to see the difference.



Balloon Project
By: Alexia Efstathiadis and Liana Rabich
MATCHED PAIRS T-TEST
Matched Pairs by Joe Rungsu
Balloon Help
Two-Sample Hypothesis Test
Assumptions:
SRS?...
Outliers?...
Skewness?...
87.12 ± 8.687
(78.434, 95.806)
BALLOON-HELP TUTORIALS!
ASSUMPTIONS
SRS? yes
Pop Normal? unknown, Proceed With Caution.
Outliers? unknown, Proceed With Caution
POP-UP BALLOON
T-Confidence Intervals
STANDARD ERROR Formula:

The Math Author Project

Here, Michael takes his At-Home Annotations to the next level. Michael used the writing tips presented in this handbook and word-processed the entire lesson on computer in a very explicit form. He tried to make it so thorough that a student who was absent could read it and fully understand and appreciate the lesson. Since you did not see this lesson, see if you can understand it from Michael's Math Author Project. Read the paper in its entirety and compare it to the original set of a class notes on this lesson.



Michael Spooner's Math Author Project

A Math Author Project for the Class Lesson on November 7

USING SLOPE AND THE FIBONACCI SEQUENCE TO
EXPLAIN AN AREA DISCREPANCY

Michael Spooner
Ms. Marra's Period 3 Class
December 7

ABSTRACT

An area puzzle was presented that involved a square and a rectangle. The square and the rectangle were each made up of two congruent trapezoids and two congruent right triangles. When the four polygons were cut from the square and rearranged, they formed the rectangle. The area of the rectangle was 1 more square unit than the area of the square. This Math Author Project explains how slope and Fibonacci numbers can be used to explain the apparent contradiction

Posing the Problem

The lesson is about a misleading area puzzle. It can be solved using slopes and the Fibonacci sequence. Here is the problem:

You are given a square dissected into two congruent right triangles, X and Y, and two congruent trapezoids. They are arranged as shown in Figure 1.

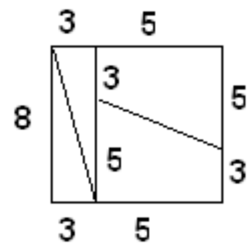


Fig. 1 The square has dimensions 8 by 8.

The same four polygons are cut from the square and rearranged to for the rectangle shown in Figure 2.

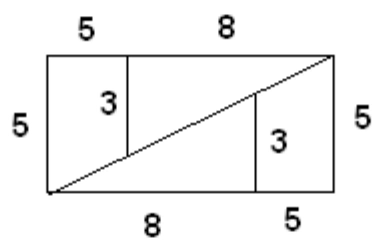


Fig. 2. The rectangle has dimensions 13 by 5.

The area of the square is 64 square units and the area of the rectangle is 65 square units. How can this happen if they are made from the same four polygons? We cannot create more area by arranging the pieces differently. Imagine if you could buy 64 square yards of carpeting and get it home and rearrange it to cover 65 square yards! We will investigate this problem using a physical scale model, algebra, coordinate geometry and a famous number sequence.

Making a Scale Model to Analyze the Problem

Are the diagrams accurate? After all, we can draw and label diagrams that cannot exist. Look at Figure 3. It shows a triangle with sides 5, 6, and 19. Recall that the triangle inequality states that the sum of any two sides of a triangle must be greater than the third side. Notice that $5 + 6 < 19$.

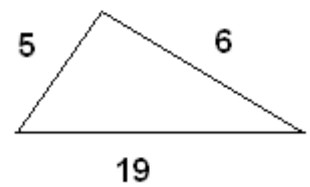


Fig. 3. This “triangle” violates the triangle inequality.

This picture is not drawn to scale, so the labeling of the sides is misleading. With this background, let’s examine the authenticity of our square and rectangle. We are going to model this problem by drawing the original square on graph paper, to be sure it is to scale. This is shown in Figure 4.

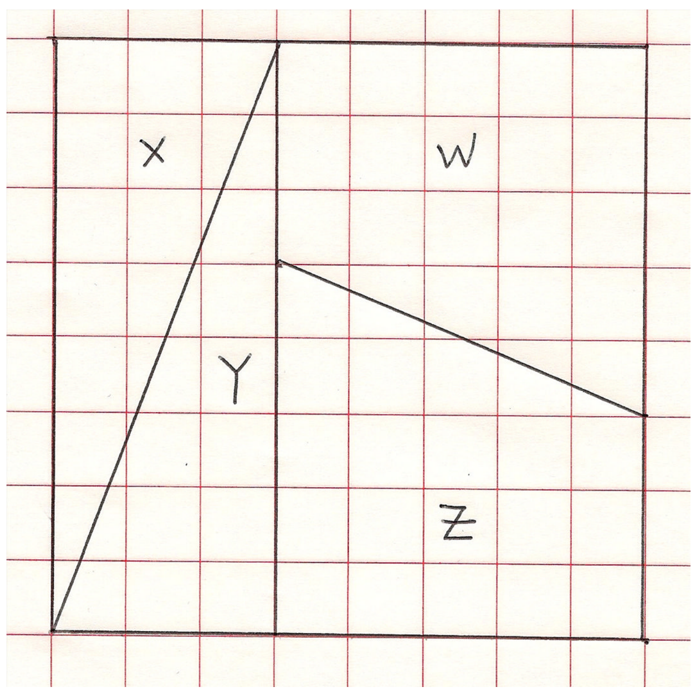


Fig. 4. All four polygons are drawn to scale.

Next, we cut the four polygons out and arrange them to form the rectangle, as shown in In Figure 5. Make allowances for imperfect scissor cuts as you look at the model.

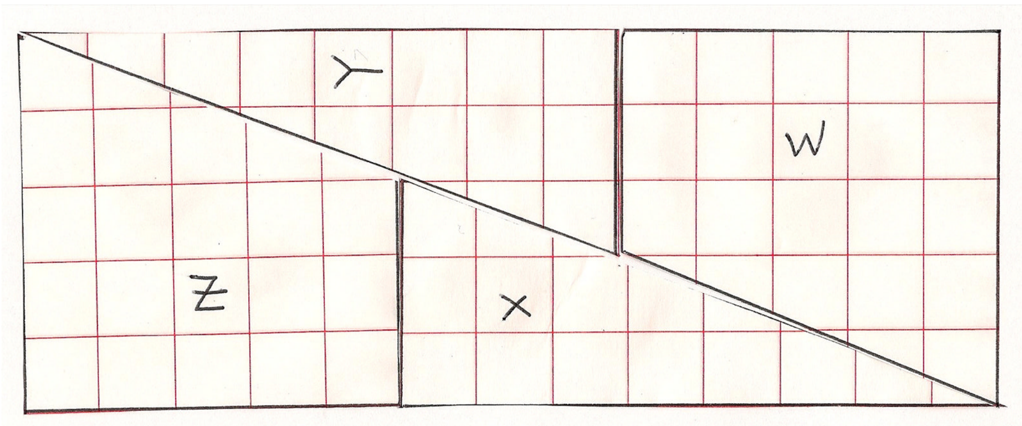


Fig 5. The scale model didn't reveal any problems!

The resulting figure does look like a 13 by 5 rectangle! The model did not help us uncover what the problem is with our discrepancy. Our mathematical intuition told us something was wrong, and not only did the model not help solve the problem, it made the problem more puzzling. We immediately realize at this point that more investigation is necessary.

This further investigation will involve the use of Fibonacci numbers and the slope of the line. You may have noticed that the three side lengths in our square are Fibonacci numbers. Since the square has congruent sides, it was necessary that the two shorter lengths equal the larger length. That is what made the Fibonacci numbers a natural fit for the side lengths.

Incorporating the Fibonacci Numbers

The first ten Fibonacci numbers are listed here:

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89\ldots$$

Each Fibonacci number can be represented using a subscript. For example, F_8 represents the eighth Fibonacci number, so we can write

$$F_8 = 21.$$

The i^{th} Fibonacci number can be expressed as F_i. We are interested in fractions of the form

$$\frac{F_i}{F_{i+2}}.$$

We will convert the fractions of this form into decimals, as shown in Table 1.

Table 1. Decimal Equivalents of Selected Fibonacci Fractions

Fraction	Decimal Equivalent
1/2	.5
1/3	.3333333…
2/5	.4
3/8	.375
5/13	.3846153…
8/21	.3809523…
13/34	.3823529…
21/55	.3818181…
34/89	.3820224

Notice that as the Fibonacci numbers increase, the fractions seem to approach a rational, repeating decimal close to 0.382. This fact will be instrumental in our solution to the area discrepancy.

We will use this newly found fact in conjunction with coordinate geometry. We will model our rectangle on the coordinate plane, as shown in Figure 6.

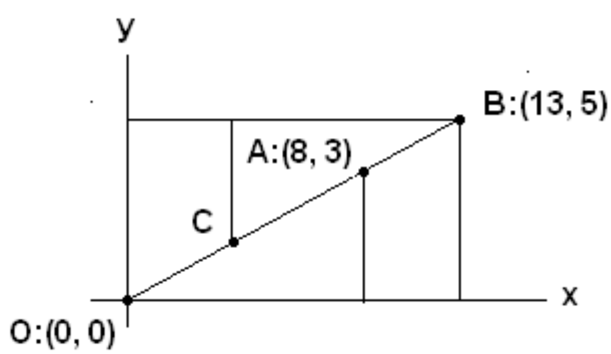


Fig. 6. The coordinates of O, A, and B are labeled.

The slope of segment OA can be found using the slope formula:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{13-8} = \frac{2}{5} = .4.$$

The slope of segment AB can also be found using the slope formula:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{13-8} = \frac{2}{5} = .4.$$

The slope of OA does not equal the slope of AB. What appears to be a diagonal of the rectangle, OB, is actually not a straight line! You may have noticed that the slopes are two of the Fibonacci fractions found in Table 1. Since the slopes of .375 and .4 are so close, the naked eye sees OB as a straight line. The actual rectangular arrangement, in an exaggerated version, is shown in Figure 7.

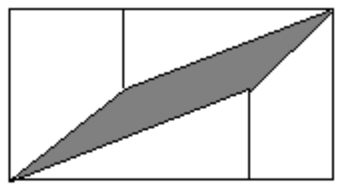


Fig. 7. The shaded space is a parallelogram with area of 1 square unit.

The extra square unit that made the rectangle have are 65 instead of 64 (like the square) is hidden in the shaded parallelogram which, when drawn to scale, is very hard to see. This experiment can be repeated with other sets of three consecutive Fibonacci numbers. As the Fibonacci numbers increase, the “kink” in the “diagonal” OB becomes even harder to see. Why? This slope that approach .382 are getting even closer to each other!

Michael’s Math Author Project is considerably longer than the class notes, because it is more thorough. It allows the writer to understand it better, and the reader to understand it better and appreciate it more. *Appreciate* a math lesson?! Let’s think about this.

Imagine if you were absent the day your class did this Fibonacci Area lesson. You would come to school the next day and ask your friend, “What did we do in math class today?” Your friend might respond, “Something about rectangles and Fibonacci numbers.”
This explanation leaves you basically clueless as to exactly what went on in the class. That’s because the topic didn’t unfurl gradually before your eyes. You got a quick synopsis. You certainly can’t understand the lesson, never mind appreciate it.
Compare this to seeing a movie. Imagine that the same day you are out sick your friends go to see the movie “Titanic.” You don’t go but you ask your friend what it is about. Your friend responds, “It’s the story of a guy and a girl on a boat that sinks.” That sentence sums it up, but it is not nearly as captivating as the three-hour movie! Do you appreciate the movie from this one-sentence explanation? Do you understand what happened in the movie?

On a mathematical level, Michael's Math Author Project is captivating, and very understandable. It teased us with a weird situation, and then did an impeccable job of explaining the situation. All of it was done gradually, in clear, logical steps. As you can see, a Math Author Project will be considerably longer than your class notes. But keep in mind that you are not striving to meet some page length requirement; the project will take on its length as a by-product of your following the writing tips given in this handbook.
Did you follow the lesson after reading Michael's Math Author Project? Can you imagine how well Michael must have understood the material to present it so clearly? As you can see, this project took time, effort and skill. You would not have time to do a Math Author Project every day.

Math Author Projects can be objectively graded. Here are some of the major criteria you can use in judging your Math Author Project. Rate your project 1 – 10 in each category, and use these guidelines to improve your project as you revise it. Keep in mind that some criteria for some lessons are not applicable.

- The title is an accurate, descriptive title for the lesson.
- The abstract is clear, succinct, and comprehensive.
- The project does a clearer job of explaining the notes than the original class notes did.
- The mathematics and the reasons are correct.
- Proofs in the lesson are explained thoroughly.
- There are sentences added to the original class notes.
- Captions for diagrams are numbered, descriptive, and formatted correctly.
- Diagrams are graduated if necessary.
- Table headings are numbered, descriptive, and formatted correctly.
- Tables are graduated if necessary.
- Color is used appropriately to help convey a mathematical idea.
- Correct mathematical terminology and notation is used throughout.
- The physical layout of the paper is of high-quality, and includes a footer with page numbers.
- Appropriate and sufficient examples are included.
- Homework examples with answers are included.
- All edits and recommended changes are incorporated.
- The depth and quality of the project are commensurate with the student's ability.

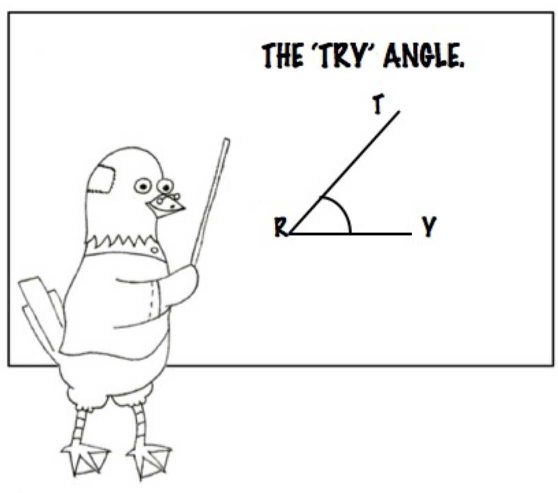
Math Author Projects are great as extra credit assignments, because they help you become more proficient in the class work you are responsible for. They will also improve your writing on standardized tests, exit exams, and Advanced Placement exams.

Imagine if each student in a class did a Math Author Project on a rotating basis for an entire school year, and the projects were copied for the entire class and kept in a binder.

You might have to do a total of 3-5 days yourself, one every few weeks over the course of the school year. You could even upload files of your notes to a common folder made available to your class. Then, everyone can obtain any day's notes electronically, whenever they need them. You would benefit from having a super set of notes covering the entire year available to you! You may want to discuss these ideas with your teacher.

Now that you are familiar with the intent of the annotation projects, you may decide to customize your project by combining features and benefits from the five different annotation projects shown.

You might be wondering, "How do I get my mathematics to *look* like Michael's Math Author Project?" Since communication is a science, we can systematically strengthen your mathematics writing skills, as you will see in Chapter 5. The key is to write, revise, and keep trying to improve your technical writing style. With practice, using this guide as a reference, you will be producing work like Michael's Math Author Project. Practice using the math you already know for starters. You will then be ready to incorporate all of your mathematics research findings into a research paper that uses the same formatting.

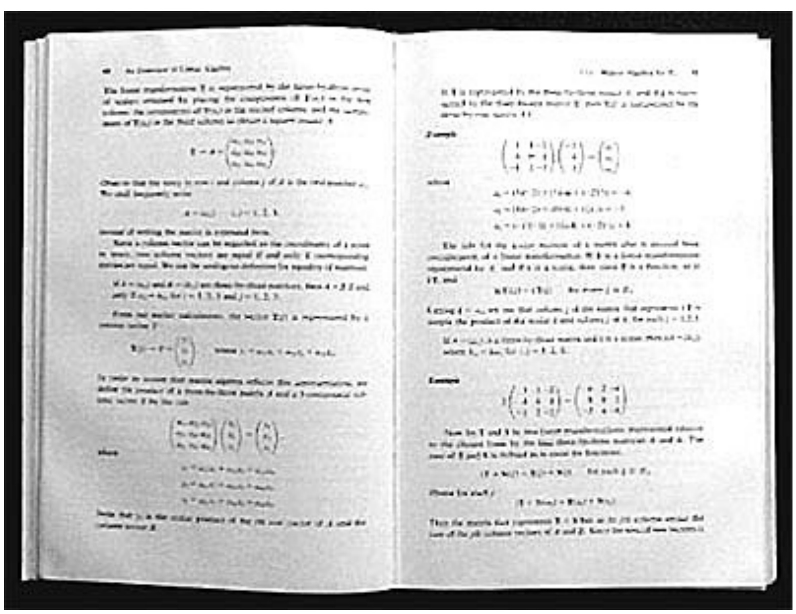


Chapter 6
Technical Writing Techniques

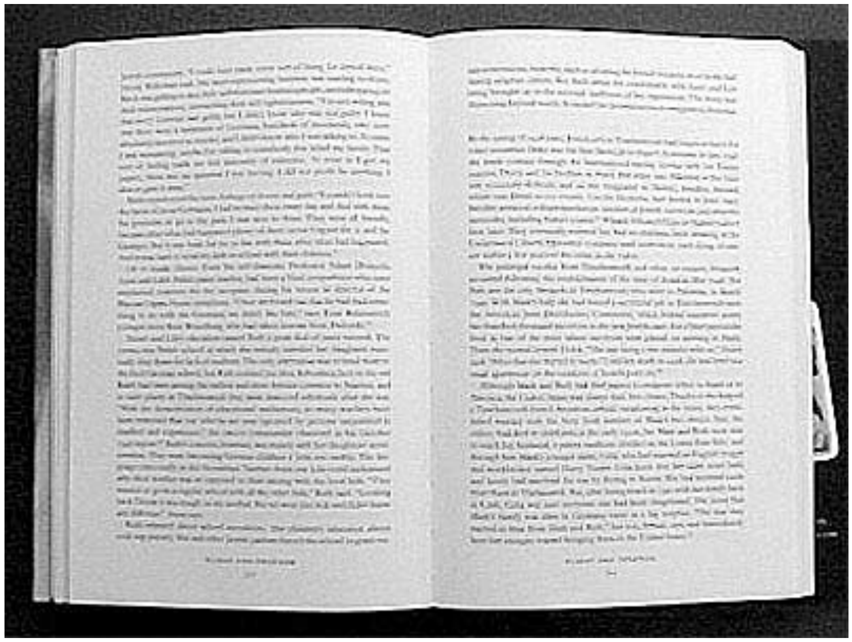
An Introduction to Technical Writing

Is mathematics writing different than other pieces you've written? Even a casual glance at a math book will yield a resounding "Yes." Try this experiment with a friend. Have your friend stand at the back of a classroom. You stay at the front. Pick up any novel. Close your eyes. Open to any page. Pick up any math textbook. Close your eyes. Open to any page. Reveal both pages to your friend from across the classroom.

Of course your friend can't read the words from that far away, but see if they can identify which book is which. Invariably they can. The novel page looks very filled with words, and the mathematics page looks "spotty" — some text is centered, some is not. There may be diagrams, some color, tables, etc.



Compare the math book's look with the look of a novel:



If you incorporate all of the writing tips in this chapter into your mathematics writing, you will create work that looks like the mathematics book. There are many tips, and they are not given in any specific order. Together, the tips show how that "spotty" math book look can be achieved. It is essential to all mathematics writing and it is not hard to learn, but does need to be practiced.

You can practice it daily by rewriting your math class notes. After you use it extensively, it becomes a habit — a good habit. In many of the tips, you will see examples of how other students have used the tips in their writing.

You can view Dylan Smith's research paper at the end of this book to get a good idea of how a student used the writing tips to become a talented technical writer. The tips are specific and well-defined. You may poke around the Internet to look for other examples of technical writing. As a newcomer, the best thing to do is to be introduced to some basic technical writing tips, see examples of them, and start to incorporate them into your writing. The following list of tips is a good way to get started.

Three-Dozen Specific Strategies for Writing Mathematics

The following technical writing tips can help you write in the format of Michael's Math Author Project. Use this section of *Writing Math Research Papers* as a reference guide.

You don't need to memorize each strategy as your read them—just know they are there and can be referred to as you write. Eventually, with enough practice, you will be writing in this form naturally, without needing the reference guide. These tips can be extended to writing up science labs, and in other disciplines as well.

Tip 1. Create an outline.

You need a skeleton of what you are going to write about. In many cases, this will be the class notes you already have. If you are writing a math research paper, you will have to organize the order of your presentation.

Tip 2. Save your work frequently.

You will be writing on a computer, using word processing software such as Microsoft Word, or Pages. There are other types of word processors as well. Make sure you save your work and back up your work by saving it a second time, on a flash drive, or by e-mailing it to your school account.

Tip 3. Use standard, academic-looking fonts.

The font called Time, or Time New Roman is the most common academic font. The text is usually written in 12-point size. Your material is supposed to look professional, sophisticated, and informative. Avoid the temptation to use fancy fonts, large letters, clip art, etc. The math you are writing is not supposed to look like a sweet sixteen invitation.

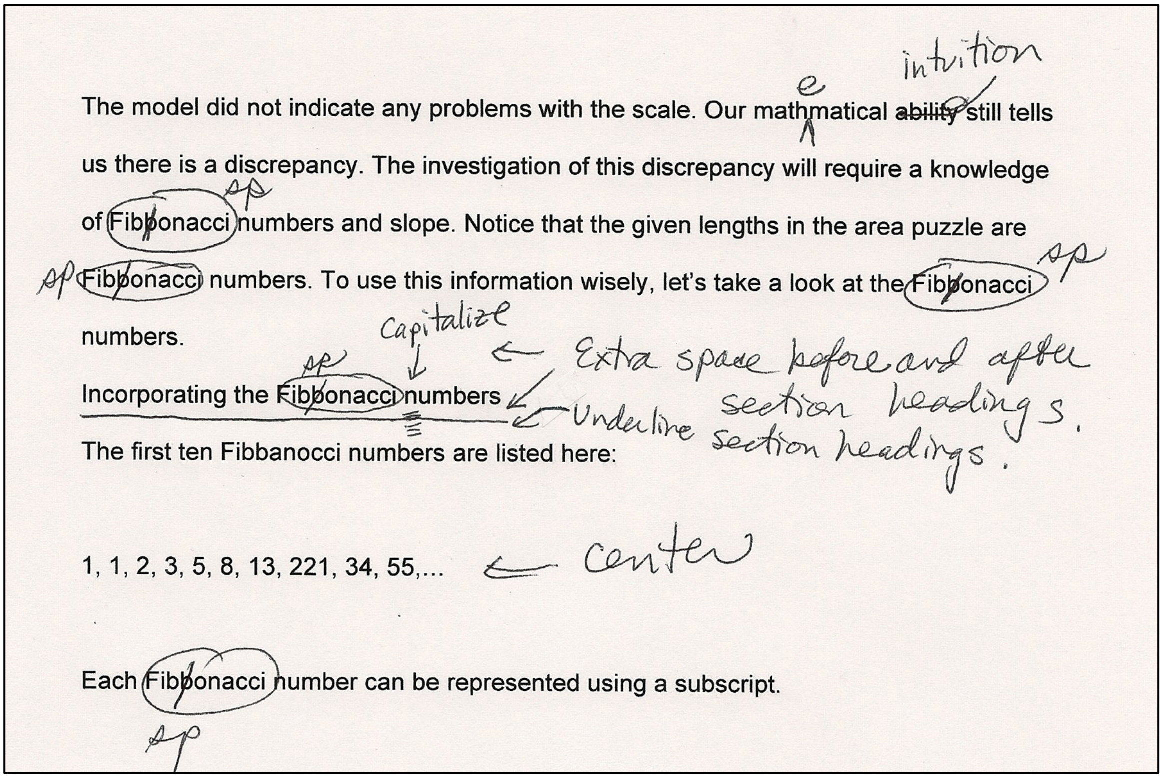
Tip 4. Number the pages in the footer.

Learn how to use the footer in your word processing software version. Number the pages from the very start of your project. This will help in the editing process. You can put your name and/or your project title in the footer as well. The footer can be 10-point. Some students put the date of the current draft in the footer and change it each time they update the project. This is helpful if the project is long term; for a short project it is unnecessary.

Tip 5. Double-space your drafts.

Double-spacing your drafts allows you, your teacher, or any other reader of your draft the chance to insert comments and make corrections right in the text where the suggested changes would occur. Single spacing does not leave enough room for effective editing. You may decide to single space your final document, depending on the requirements set by the teacher who assigned it. When you go to college, you will learn specific writing style specifics that your school prefers.

The following example shows how well editing can fit in between the lines of double spaced text.



The model did not indicate any problems with the scale. Our mathmatical ~~ability~~ still tells us there is a discrepancy. The investigation of this discrepancy will require a knowledge of Fibbonacci numbers and slope. Notice that the given lengths in the area puzzle are Fibbonacci numbers. To use this information wisely, let's take a look at the Fibbonacci numbers.

Incorporating the Fibbonacci numbers

The first ten Fibbanocci numbers are listed here:

1, 1, 2, 3, 5, 8, 13, 221, 34, 55,...

Each Fibbonacci number can be represented using a subscript.

Tip 6. Use an equation editor to create mathematical expressions.

Most word processing programs have built-in equation writing features. These equation editors allow writers of mathematics to create professional-looking equations. The variables are automatically italicized, as they should be. The equation editors usually have very user-friendly toolbars and menus so they are not difficult to master. The following is from Allison's research paper on pairs of triangles that have the same area, same perimeter, yet are not congruent.

$$a = \frac{-(2s-c) \pm \sqrt{(2s-c)^2 - 4(-1)\left(-s^2 + sc - \frac{A^2}{s(s-c)}\right)}}{2(-1)}$$

Notice how square roots and fractions look just like they would in a math textbook!

An equation in Alexandra's paper on escribed circles has fractions in parentheses. Notice how the parentheses enlarge to the height of the fraction.

$$r = \left(\frac{a + b - c}{2}\right)$$

If you have a fraction in the numerator, denominator, or both, equation editors can easily handle it. Take a look at a fractional equation from Mike's paper on the eccentricity of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{\frac{c^2(1 - e^2)}{e^2}} = 1$$

Diana's work with compound interest required an extensive use of nested parentheses, subscripts and exponents, which is shown here.

$$B_4 = \left(p\left(1 + \frac{0.06}{4}\right)^3\right) + \left(p\left(1 + \frac{0.06}{4}\right)^3\right) \times \frac{0.06}{4}$$

Equation editors have a text mode in which you can type standard English in conjunction with mathematical expressions, as this line from Erin's paper shows. The words and the mathematics can both be written in equation editor.

$$\text{Area of the yellow triangle} = \tfrac{1}{2}bc\sin(\alpha + \beta)$$

You can embed equation edited expressions in the midst of a sentence, as Molly did here. The sentences were lengthy, so they were typed in a word processing program and the mathematical expressions were inserted. Notice the use of absolute value and the Greek alphabet.

> An ε -neighborhood of 3 should map within a ε –neighborhood of 17. Now we are going to show that if $|x - 3| < \frac{\varepsilon}{7}$, then $|f(x) - L| < \varepsilon$.

When you have a series of centered digestion algebraic digestion points separated by left-justified reasons for each step, insert your equations separately. It is easier to make editorial changes if they are separate insertions from equation editor. You can type the left justified reasons in word. Take a look at Steven's paper on perfect triangles (triangles with numerically equivalent area and perimeter) and take note that each equation was pasted up separately.

Now suppose b is an even number:

$$a = \frac{b^3 + 16b}{2b^2 - 32}$$

Change the equations by separating 16b into -16b + 32b.

$$a = \frac{b^3 - 16b + 32b}{2b^2 - 32}$$

Separate into two fractions.

$$a = \frac{b^3 - 16b}{2b^2 - 32} + \frac{32b}{2b^2 - 32}$$

On the left side of the equation factor out b from the numerator and 2 from the denominator.

$$a = \frac{b(b^2 - 16)}{2(b^2 - 16)} + \frac{32b}{2b^2 - 32}$$

Simplify the first fraction.

$$a = \frac{b}{2} + \frac{32b}{2b^2 - 32}$$

Most of the algebraic issues you will need to tackle have been covered in the examples above. Geometrical notation is also easy with equation editor. Take a look at how common geometric notation is available on your equation editor.

AB||DE, AB⊥AC, EF⊥DE and BF≅DC.

Familiarize yourself with the features of your computer's equation editor. Just try to copy the equations above which used many features of the equation editor.

Tip 7. Format your mathematics like a technical writer.

This tip is the one that will get your mathematics documents to have that "textbook" look--the "spotty" look that makes it look nothing like a novel. As mathematical ideas are developed, mathematical expressions need to be “digested” by the reader. It is mandatory that you draw attention to each step of a development so the reader can follow your work, check in their mind that they understand, and pause before continuing. Algebraic and geometrical results that need to bee digested will be centered, and appear on their own line. We call these **digestion points**. The reasons for each centered step must be given before the step, and these reasons are left justified, on their own line. This format, combined with double–spacing, can make mathematics manuscripts quickly grow in page length. Notice how Michael’s Math Author Project was over six pages long, while the original class notes took less than one page.

Let’s examine one example of the formatting here. The following paragraph is written as if it was part of a novel; the person writing is not using correct technical writing skills.

> To change the equation of the line 2x - 3y = 24 into the slope-intercept form of the equation of a line, recall that slope-intercept form is y = mx + b. Begin to isolate y by subtracting 2x from both sides of the equation. Multiply both sides by –1. This yields 3y = 2x + 24. Next, divide both sides of the equation by 3. The result is y = 2/3x - 8. The coefficient of x, 2/3, is the slope and –8 is the y-intercept.

Now let’s look at virtually the exact same paragraph rewritten using the writing tips we’ve explored thus far. Notice that the reasons can often be stated in the imperative, making them short and to the point.

> To change the equation of the line 2x - 3y = 24 into the slope-intercept form of the equation of a line, recall that slope intercept form is
>
> $$y = mx + b.$$
>
> Begin to isolate y by subtracting 2x from both sides of the equation.
>
> $$-3y = -2x - 24$$
>
> Multiply both sides by –1.

$$3y = 2x - 24$$

Divide both sides by 3. $m = \frac{2}{3}$

$$y = \frac{2}{3}x - 8$$

The coefficient of x is the slope, m.

$$m = \frac{2}{3}$$

The constant term is the y-intercept, –8.

$$b = -8.$$

Notice how clear the centered algebraic expressions and the left–justified reasons make this development. You would appreciate this even more if you were reading about more difficult mathematics. Read through the examples in Chapter 11 to get an idea of how this style makes reading mathematics easier.

Let's look at an excerpt from a geometry proof that rewrites a "Statements–Reasons" proof in our new form. The "Statement-Reasons" column proof format was created to make proof more understandable for students. It is not how mathematicians present their proofs. Take a look at the format of the proof below. Sometimes this is called a paragraph proof.

We are given that

$$AB \| DE,\ AB \perp AC,\ EF \perp DE \text{ and } BF \cong DC.$$

We want to prove that

$$\Delta ABC \cong \Delta DEF.$$

Since $AB \perp AC$ and $EF \perp DE$, we know that

$\angle A$, $\angle E$ are right angles.

Since all right angles are congruent,

$$\angle A \cong \angle E.$$

If equals are added to equals, the results are equal, so

$$BC \cong FD.$$

Lines AB and DE are parallel to each other, and are cut by transversal BD, forming alternate interior angles B and D. Therefore,

$$\angle B \cong \angle D.$$

If two triangles have two pairs of corresponding angles congruent, and the included sides are also congruent, the triangles are congruent, so

$$\Delta ABC \cong \Delta DEF.$$

The paragraph form of a geometric proof is different than the "Statements–Reasons" form, and mimics the way real mathematicians do proofs more authentically. Notice that the left–justified reasons are given before the centered algebraic or geometric result. Check out the geometry proofs in Dylan's paper at the end of this book. As you can see, this new writing technique will have profound influence on the mathematics you write.

Tip 8. You can "wrap" your left justified reasons and explanations.

The reasons and explanations that precede the digestion points can take up more than one line. In fact, the reasons can be sentences or even paragraphs long. Notice the reasons in the examples from Tip 7, and also take a careful look Chapter 11. The student examples will make this concept very clear to you.

Tip 9. Do not "wrap" your centered algebraic expressions.

It is harder for the reader to follow equations that take up two lines, especially as they are algebraically manipulated throughout the development. If an equation is too long, consider having just that one page in your document in the landscape page set up position. This can be done on any word processor. Be careful not to landscape the whole document, just the pages with lengthy equations.

Tip 10. Number any equation that will be referred to later.

If you need to refer to one of your centered algebraic equations, you can name it with an Equation number. This is only necessary if you need to refer to the equation in other parts of your writing, and you need a way to identify it, since there are so many equations in your manuscript. In parentheses, next to the equation, right justified, you can put "Equation" and the number, or just the number. The following example is from Jeanne's paper on Heron's Formula. Later on, in the paper, she can refer to "Equation 7 on page 14" and the reader will know exactly where the equation came from.

$$h^2 = \left(\frac{(2(s-b))(2(s-c))(2s)(2(s-a))}{4b^2} \right) \qquad \text{(Equation 7)}$$

To see more examples of this, look in Chapter 11.

Tip 11. All figures must have a figure number and a descriptive, one-sentence caption.

The caption should give information about the diagram, not just be a "nameplate." To make an analogy, if you pasted up a picture of the Eiffel Tower into your paper, you would not write "Eiffel Tower" as the caption.

Fig. 1. The Eiffel Tower.

This caption is not descriptive enough, and it is not a complete sentence. You should be imparting more information. Look at the following possible descriptive captions.

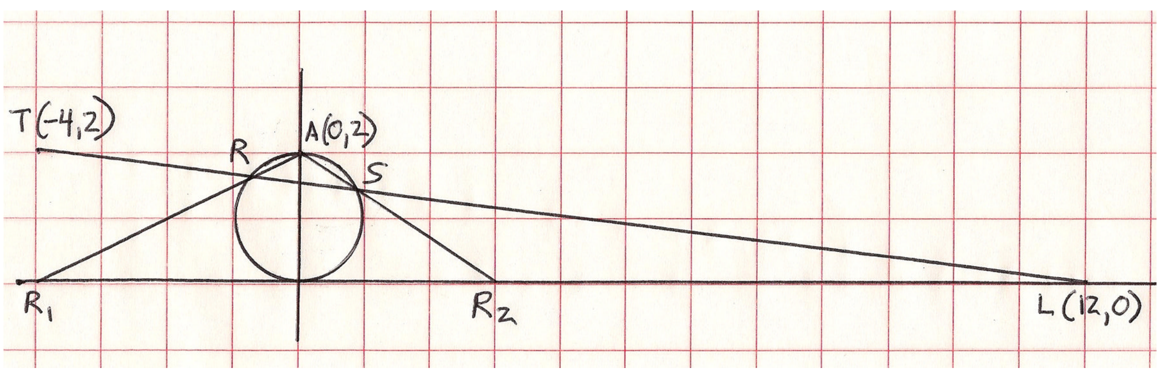
Fig. 1. The Eiffel Tower was the entrance to the 1889 World's Fair in Paris.
Fig. 1. The Eiffel Tower was the tallest structure in the world until 1930.
Fig 1. At 1,063 feet tall, the Eiffel Tower is the tallest structure in Paris.

Don't make your caption so long that it wraps around to a second line. Those longer thoughts can be written right in the text, and you can think of a shorter caption. Refer to your figure numbers as you explain material in the text. Don't say, "In the figure below the median meets the opposite side at P." Instead, say, "In Fig. 7 the median meets the opposite side at P." Look at Dylan's paper for more examples of captions.

Tip 12. There are several ways to display graphs.

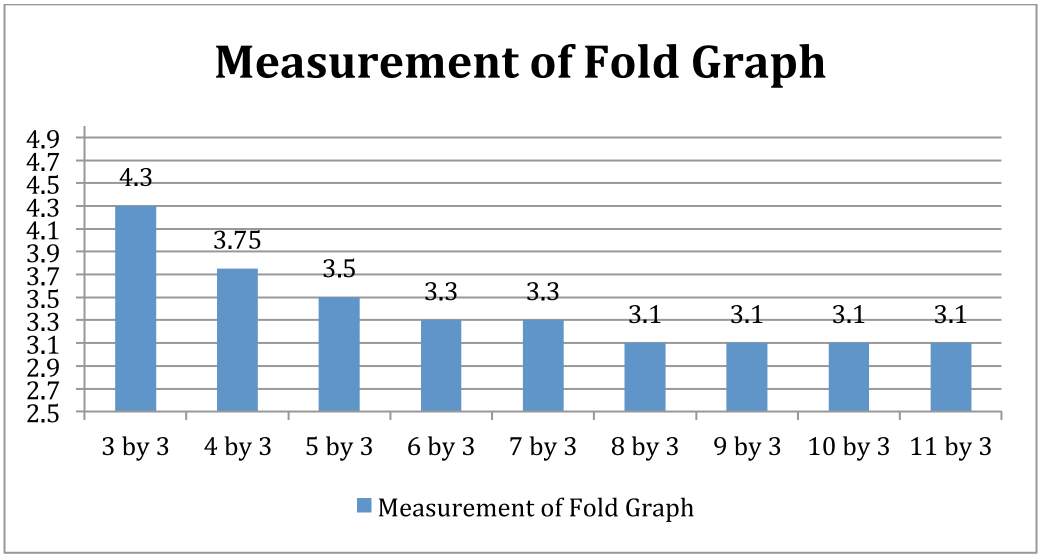
You will often need to display graphs in your mathematics writing. There are several ways to display graphs.

If you need to draw the graph by hand, you can do so, and then just scan your graph. Sunaina's work on solving quadratic equations using a unit circle with center at (0, 1) required she make a graph using compass and straightedge to check out some mathematical properties.

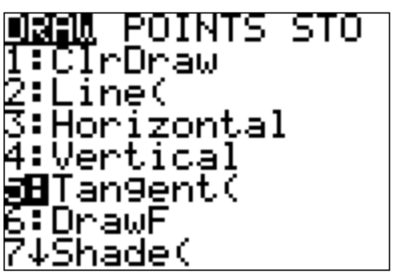


Hand-drawn graphs allow you the ability to customize, color, and include all idiosyncrasies in the exact style and format you want. You can import your scan into a drawing program and further annotate or customize it.

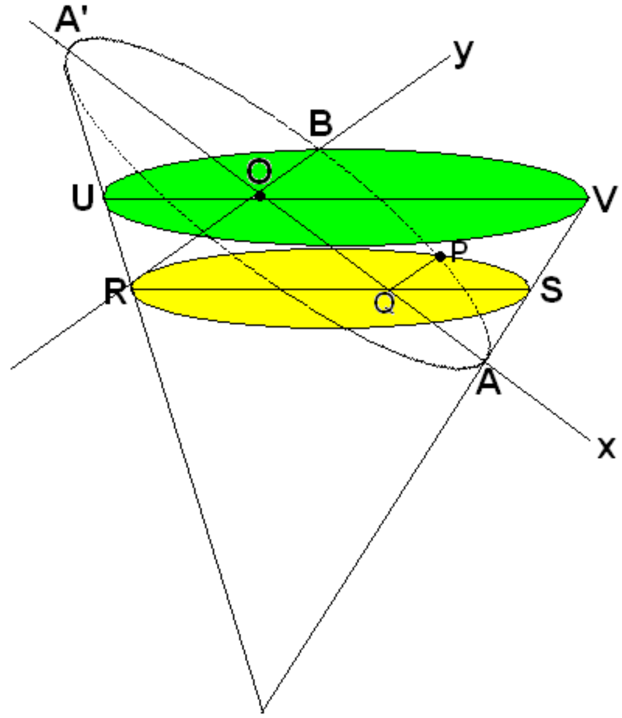
Andrew did a paper on measuring the length of the crease when a one pair of a rectangular index card's opposite vertices are folded onto each other. He did a summary graph of his data using Excel.



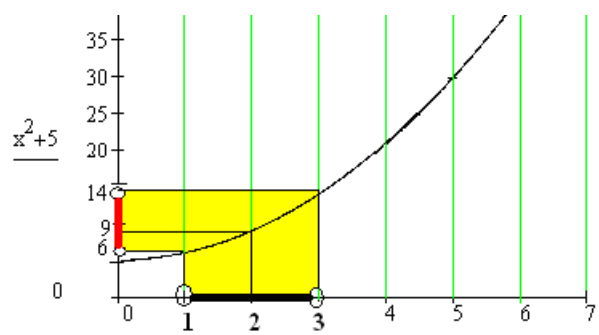
You can print calculator screens using software available (usually as a free download) form the calculator's manufacturer. Below is a graph of a tangent line to a parabola from Cecilia's calculator. If necessary, you can paste the screen shot into a drawing program to add other material to the screen shot. Notice how you can change the size of the screen shot if you need to illustrate a specific point.



The following 3-dimensional graph from Michael's paper on ellipses was drawn entirely using a drawing program. No graphing software was involved.



If you create a graph using a graphing program, you can still paste it into a drawing program to add custom touches to your graph. Julianne's paper on limits featured several graphs she created using a graphing program. She then pasted the graphs into a drawing program to add custom touches she needed that the graphing software could not do, as shown below.



Keep in mind that all of your graphs are considered figures, and need a figure number and a descriptive caption.

Tip 13. Use the words "We", "They", and "Let's."

As you are writing, use the words "we," "they," and "let's." These words invite the reader into the paper, and subconsciously convey that the author and the reader are in this together. Try to avoid using "you" and "I." Keep this in mind if you ever do an oral presentation on your work. It is an appropriate style for high school students.

Tip 14. Don't use judgmental phrases.

Don't say something is "easy," "simple," "obvious," or "clear," because that doesn't help the reader understand the information. In fact, if the point you claim is not "easy" for them, they are now intimidated. A concept obvious to one person may not be obvious to another. The judgmental phrase doesn't help the reader digest the mathematics, so it should be avoided.

Tip 15. Separate your manuscript In to different sections.

Your manuscript should be divided into different sections at logical junctures. You have probably read (and maybe even written!) run-on sentences in your school career. You realize that a run-on sentence contains too much information and is too long. Separating the sentence into several shorter sentences is usually helpful. If you expand this idea to a whole paper, imagine what a run-on paper would be like!

A paper needs reflection and digestion points, places where someone can pause. The reader also needs to be signaled when different aspects of the work are being covered. Dividing your paper into sections and giving each section a title is necessary on a lengthy manuscript. A short piece would need fewer sections. Look at Michael Spooner's Math Author Project, and Dylan's paper, and see how they divided the documents into sections.

You need a standard hierarchy of how your sections subdivide your work. Each type of subsection is written differently, so it clearly defines how the paper is being divided. Look at the following hierarchy and then we will explain it. Imagine that there is text in between each section heading; we're just looking at the section headings here.

INTRODUCING THE COMPLEX NUMBERS

I. The Real Number System

<u>A. The Set of Rational Numbers</u>

The Set of Integers

The Set of Whole Numbers

The Set of Natural Numbers

<u>B. The Set of Irrational Numbers</u>

II. The Set of Imaginary Numbers

Notice that the paper is on Complex Numbers. This main topic title is capitalized. There is a section on Real Numbers, and Real Numbers are part of Complex Numbers, so the Real Number title must be written differently; it is left-justified and bolded. The Set of Rational Numbers is a subsection of the Real Numbers section, so it must be written differently; it is left-justified, not bolded, and underlined.

The next three titles are again subsections of the one above it, so they are shown with a different style. They are centered, not bolded, and not underlined. The Set of Irrational Numbers title is written in the same style as The Set of Rational Numbers. Therefore, it is not a subsection of Rational Numbers, however, it is a subsection of The Real Number System.

The Set of Imaginary Numbers is written in the same style as The Real Number System title. So, it is not a subsection of the Reals, but its own section with the same formatting as the Reals. You can tell that it is not a subsection of The Set of Irrationals since the style was used already previously in the hierarchy.

Keep in mind that you should leave an extra space before and after a section heading. This makes it stand out to the reader's eye.

The above example of a hierarchy is not the only one. You can adapt yours using different combinations of justification, capitalization, bold, underlining, and even color if you are writing an electronic document or can print your paper in color. Don't go overboard though.

The key is to be consistent. In all cases, do not have a subsection heading alone at the bottom of a page. There must be at least one line of text before the page breaks. If a section heading appears at the bottom of a page, insert your own page break and have it appear at the top of the next page.

The beauty of the word processor is that you can insert, move, or change subsections (and you will!) when you proofread your drafts.

Tip 16. Use segues at the beginning and end of each subsection.

Make sure you have segues, which are transition sentences, at the beginning and/or end of your subsections. Don't have the presence of the next section come as a surprise to the reader. Prepare them; have them expect that it is coming and explain why it is placed there. This keeps the flow smooth. Every section should have an introductory beginning sentence and a concluding ending sentence. Look at the first and last sentences of each section of Michael's Math Author Project, and Dylan's paper, and look for these transition sentences.

Tip 17. Create a descriptive title for your work.

This title may be longer than you are used to. You may think of a title and then revise it. It may be best to write it after you finish the project, when you best understand its main points. The title of Michael Spooner's notes was

USING SLOPE AND THE FIBONACCI SEQUENCE TO SOLVE
AN AREA DISCREPANCY

These titles would not have been as informative:

SQUARES AND RECTANGLES

AN AREA PUZZLE

Avoid "cute" titles, such as:

FIBONACCI MEETS A DILEMMA

A PUZZLING PUZZLE PARADOX

Take a look at the title of Dylan's paper in the back of the book. Do not be concerned that you may not understand the content implied from the titles of the papers you look at; just try to get the flavor of the explicit nature and the length of the title.

Tip 18. Make an academic–looking, sophisticated, informative cover page.

Use the same 8.5 x 11 in. paper you used for the paper. Use the same font and the same point size. Include the descriptive title, centered, single–spaced, in inverted pyramid form. This means that each line is shorter than the previous line. Also include your name, grade, teacher, school, and contact information (address, e-mail, phone number). Do not use color for color's sake, and do not use construction paper, markers, etc., like you might have for a middle school report on Abraham Lincoln. Look at Michael's cover page for his Math Author Project or Dylan's title page for his paper.

Tip 19. Write an abstract on the page after your cover page.

An abstract is a summary of the paper. Although it appears first, right after the cover page, it is actually written last, when you are in the best position to summarize your work. An abstract is a short synopsis that someone can use to decide if they want to read the entire paper. In college, you will use abstracts often in the library. You can look up abstracts for many topics on the Internet, and decide if you want to download, or even purchase, the document. It would be difficult to tell from just the title if you wanted to get the entire document. When you are given an assignment in any subject, you will read many abstracts, and based on the abstracts, decide which books or articles you need to read. Abstracts are short and to the point; they don't "develop" concepts. They state the topic, summarize, and give results. Here is an abstract from Michael's research paper on the eccentricity of an ellipse:

> This paper combines properties of ellipses and limits to evaluate the graphs of the ellipses as their eccentricities approach 1. Limits are explored in a multitude of ways: numerically, graphically and geometrically. The purpose of this is to evaluate what happens to an ellipse when its eccentricity approaches 1. In the limit, the ellipses with increasing eccentricities become more oblong and the "ellipses" with eccentricities equal to 1 are actually parabolas. Recommendations for further research include taking the limits of hyperbolas as their eccentricities approach 1.

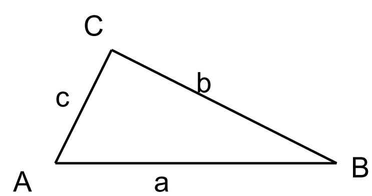
Also look at Michael's abstract on his Math Author Project, and Dylan's abstract in his research paper at the end of *Write On! Math*.

Tip 20. Label geometric diagrams properly.

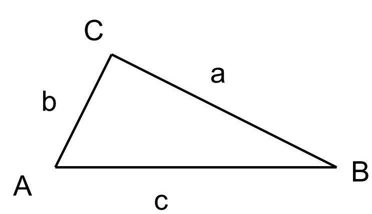
Remember that in a geometric diagram, capital letters stand for points and lower case letters stand for segments. If you needed to start a sentence with a lower case 'c', changing it to a capital 'C' is not only incorrect, it changes the meaning. There are many cases in geometry where there are capital and small letters in the same problem.

Traditionally, when you label a triangle, the sides opposite each angle are named with the same letter, only in lower case.

Incorrect:



Correct:



Carefully note your use of the symbols for line segments, lines, arcs, angles, congruence, similarity, and inequalities. These are all found in equation editors. See Dylan's paper in the back of this book for more examples of properly labeled diagrams.

Tip 21. Don't start a sentence with a variable, numeral or symbol.

Often when you are citing data from a table in your work, you might be tempted to start the sentence with a numeral. Keep in mind that the sentence can always be reworded to avoid this.

> Incorrect: "c is the side opposite angle C."
> Correct: "The side opposite angle C is c."
>
> Incorrect: "2 is the only even prime number."
> Correct: "The only even prime number is 2."
>
> Incorrect: "173 is the last entry in Table 12."
> Correct: "The last entry in Table 12 is 173."

Tip 22. Use an Appendix for excessive amounts of data.

If a table runs more than two pages, it is often not put in with the text; it can be placed in an Appendix. The Appendix is in the back of the paper. Excessive amounts of data from statistical analyses or created from spreadsheet programs, are often put in the Appendix. This is material that, if inserted in the manuscript, would be so long that it would disrupt the flow and continuity of the writing.

The specific numbers are not needed to understand the work as the reader reads it. The data is included in the Appendix so the reader can look at it, verify it, search for patterns or trends, etc. It makes the work complete and professional, without upsetting the writing flow. You may also decide to keep excessive amounts of data in a file readers can access with a link you include in your paper.

Many documents don't need an Appendix, just keep this feature in mind if yours does. You may let readers know in your abstract that you have included an appendix.

Tip 23. Use correct mathematical terminology.

Don't use mathematical slang, which may be acceptable when you are discussing something in class.

> Incorrect: "The thing next to the x is 18."
> Correct: "The coefficient of x is 18."

Incorrect: “Cross multiply.”
Correct: “Find the product of the means and extremes.”

Incorrect: “FOIL”.
Correct: “Multiply the two binomials.”

Incorrect: “Move around the terms.”
Correct: “Commute the terms.”

Using the correct mathematical terminology is a crucial requirement for achieving precision in your work. If you are ever in doubt about the sophistication level of a phrase you are considering using, ask your teacher or math department chairperson.

Tip 24. Carefully note how you use pronouns.

You might have learned in English that it is not good to repeat the same word too often. If you were writing a paper on the Kennedy assassination, you might have a sentence like this:

> “Kennedy was shot in Dallas during a Presidential motorcade in which Kennedy was driving in a convertible with the top down.”

Most likely, you would use a pronoun to take the place of the noun “Kennedy:”

> “Kennedy was shot in Dallas during a Presidential motorcade in which he was driving in a convertible with the top down.”

Remember that in technical writing, clarity is the goal. Any “jazzed up” writing that sacrifices clarity is counterproductive. In particular, watch your use of pronouns to replace nouns. Look at the following example from a student paper on how hyperactive children react to educational software.

> “The kids sat still and were mesmerized by the entire set of DVDs. We were surprised that they were so good.”

Were the people surprised that the children were so good, or that the DVDs were so good because they engaged the children and the children sat still? This is a key point; are they commenting on the kids or the DVDs? The word ‘they’ is a pronoun that replaces a noun. The noun it replaces is called the antecedent of the pronoun. In this case we can’t tell what the antecedent is, the DVDs or the kids. Therefore, the expense of using ‘they’ instead of repeating “the kids” or “the DVD’s” was enormous; it sabotaged the intended meaning.

Tip 25. Rewrite entire equations as you manipulate them.

When you are working on an algebraic development, on each step you concentrate on a different part of the equation. Look at the following equations from Meghan's paper on ellipses, and notice that the entire equation is written on every line. Explanatory sentences in between the equations refer to the changes made in it. You may decide to highlight where the changes are to help the reader.

Square both sides of the equation.

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2$$

Square the underlined terms.

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + x^2 - 2cx + c^2 + y^2$$

Subtract x^2, c^2, and y^2 from both sides of the equation.

$$2cx = 4a^2 - 4a\sqrt{(x-c)^2+y^2} - 2cx$$

Subtract 2cx from both sides of the equation.

$$4a\sqrt{(x-c)^2+y^2} = 4a^2 - 4cx$$

Divide both sides of the equation by 4.

$$a\sqrt{(x-c)^2+y^2} = a^2 - cx$$

Square both sides of the equation.

$$a^2 (x-c)^2 + a^2 y^2 = a^4 - 2a^2cx + c^2 x^2$$

Square (x-c)

$$a^2 (x^2-2cx+c^2) + a^2 y^2 = a^4 - 2a^2cx + c^2 x^2$$

Distribute a^2 over $(x^2-2cx+c^2)$.

$$a^2 x^2 - 2a^2 cx + a^2c^2 + a^2y^2 = a^4 - 2a^2cx + c^2 x^2$$

If the entire equation was not reprinted, it would have become difficult for the reader to constantly switch back and forth from the equation and the separate dissected parts. If ever a manipulation of an equation requires a lengthy explanation, remember that reasons can take as many lines as possible; a reason could be its own paragraph.

Tip 26. A table must be centered on the page.

All tables have a table number and a descriptive title. The table number and title are centered above the table and underlined. Single–space your tables. Do not write on the side of a table, in the margin, like a newspaper might. Look at Olivia's table below.

Table 13. Area of One-Inch Squares

Number of Divisions	Graph Paper	Interior Points	Boundary Points	Area by Pick's	Actual Area
1	1"	0	4	1	1
2	1/2"	1	8	4	1
4	1/4"	9	16	16	1
8	1/8 "	49	32	64	1
16	1/16"	225	64	256	1

You should not split a table over a page break. You will need to move the table so it stays on one page. You can highlight rows or columns in color if that helps convey a mathematical point.

If a table has many rows and only a few columns, consider presenting it in a horizontal fashion, as Danielle did while she was generating data to look at patterns.

Table 3. Values of the Equation $f(x) = x^2 - x + 41$

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
f(x)	41	41	43	47	53	61	71	83	97	113	131	151	173	197

Imagine how long and thin this table would have been if it was presented vertically. See Dylan's paper to see some more examples of table titles.

Tip 27. Develop complicated tables gradually, one step at a time.

Just as you can present diagrams in a gradual fashion, you can present tables in a graduated fashion, too. Again, books have space limitations. Often tables have many columns and/or rows that evolved one at a time. If you need to describe such a table, present it in stages. Show the first columns and rows that begin the discussion in the first table, and then add columns and rows in subsequent tables. Each table must have a table number and a descriptive title, as usual. You should add explanatory sentences between each graph to explain what is transpiring as the columns are annexed. Here is an example of graduated tables from Gabrielle's research on forming Pythagorean triples from Fibonacci numbers. Gabrielle's explanations in between each table are not included—the idea is for you to see how columns and color-coding have been added a little bit at a time. The explanations she gave made it easier for the reader to understand the message. Imagine being presented with the final table only and trying to understand what the significance of all the entries.

Gabrielle's first table has only two columns, so the reader can digest and process it before tackling more columns.

Table 6. Variable Representation of Fibonacci Numbers

Number of Term	Fibonacci Number
1	1
2	1
3	2
4	3
…	…
n	a
n+1	b
n+2	a+b
n+3	a+2b
…	…

The reader can totally focus on the addition of Gabrielle's third column because the first two columns have already been explained. A reader who had trouble deciphering the first two columns would not even move on to the three-column table.

Table 7. Progressive Fibonacci Numbers

Number of Term	Fibonacci Number	Deconstructed Fibonacci Numbers
1	1	…
2	1	
3	2	
4	3	
…	…	
n	a	
n+1	b	
n+2	a+b	a + b
n+3	a+2b	b + a +b
…	2a + 3b	b + 2a + 2b
	3a + 5b	2b + 3a + 3b
	5a + 8b	3b + 5a + 5b
	8a + 13b	5b + 8a + 8b

This table adds a fourth column that builds off of the third columns, as Gabrielle explained in her test. As the author, you are trying to "hold the hand" of the reader as they navigate through the series of tables.

Table 8. Factoring Out Fibonacci Numbers

Number of Term	Fibonacci Number	Deconstructed Fibonacci Numbers	Factoring
1	1	…	
2	1		
3	2		
4	3		
…	…		
n	a		
n+1	b		
n+2	a+b	a + b	**1** (a + b)
n+3	a+2b	b + a +b	**1**b +**1** (a + b)
…	2a + 3b	b + 2a + 2b	**1**b + **2** (a + b)
	3a + 5b	2b + 3a + 3b	**2**b + **3** (a + b)
	5a + 8b	3b + 5a + 5b	**3**b + **5** (a + b)
	8a + 13b	5b + 8a + 8b	
…	…	…	
			$b \cdot b + (a+b)(a+b)$

A reader knows to make sure they care clear on the first four columns before looking at the addition of the fifth column.

This is the final table. Imagine it being the first table that was presented. It would be quite difficult to understand. The graduated tables, along with explanations between each, maximized the chances of the reader fully comprehending the meaning inherent in the tables.

Table 9. Categorizing Fibonacci Numbers

Number of Term	Fibonacci Number	Deconstructed Fibonacci Numbers	Factoring	Categorizing
1	1	…		
2	1			
3	2			
4	3			
…	…			n+ 2 terms
n	a			
n+1	b			
n+2	a+b	a + b	**1** (a + b)	
n+3	a+2b	b + a +b	**1**b +**1** (a + b)	
…	2a + 3b	b + 2a + 2b	**1**b + **2** (a + b)	
	3a + 5b	2b + 3a + 3b	**2**b + **3** (a + b)	
	5a + 8b	3b + 5a + 5b	**3**b + **5** (a + b)	n+1 terms
	8a + 13b	5b + 8a + 8b	**5**b + **8** (a + b)	
…	…	…	…	
2n+3			$b \cdot b + (a+b)(a+b)$	

You will never have a table follow another table without some explanatory material in between them. A table will never end a section. There must be some closure; some reference to what the table displayed.

Tip 28. Use color creatively, but with discretion.

You can use color to highlight. Color is great to help the reader differentiate between parts in a diagram. Color can be used to highlight entries in tables and parts of algebraic equations. If necessary, you can coordinate items by color. This means you are doing all related items in the same color to help the reader understand a point you want to make.

You can achieve the color by coloring the font, by using your word processor's highlighter, or by underlining different parts in different colors. You can make references to the color-coding in your writing. Do not use color just to "dress up" your document--keep it professional and tasteful and use color with care, creativity and discretion.

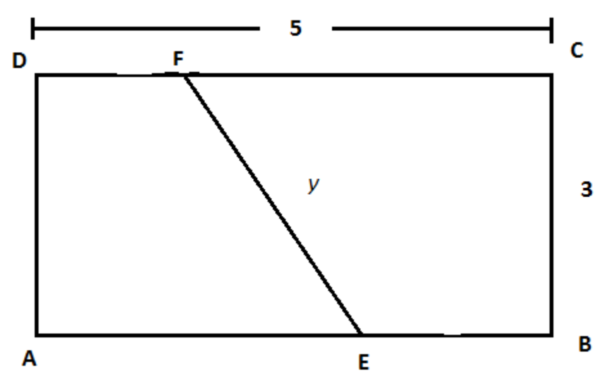
Tip 29. Use a graduated series of diagrams to develop an intricate diagram.

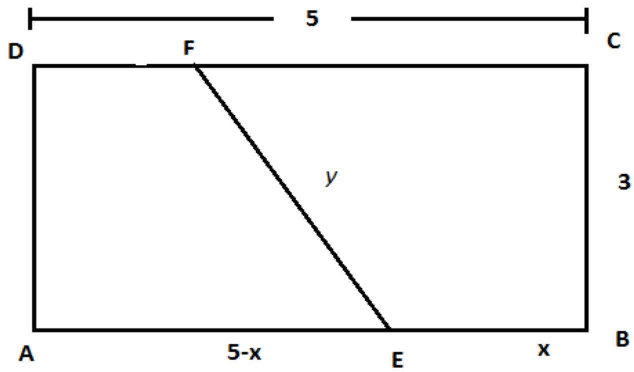
Often in mathematics, especially in geometry, there are diagrams that get rather intricate. Many textbooks and math journals have space limitations and, as a result, cram all of the information into one single diagram. The reader doesn’t know in what order the diagrams were changed, as lines were added, etc. The reader can’t tell how the diagram evolved, and this is crucial to the development of the topic.

When you encounter this situation in your mathematics writing, plan on using a graduated series of diagrams. This is a set of diagrams that dissects an intricate diagram into its components, one diagram at a time, in the order the diagram evolved.

One diagram from your class notes or textbook could turn into two or more diagrams when you write your manuscript. Each diagram in the graduated series of diagrams builds upon the previous diagram. It includes the previous diagram and adds new information, one piece at a time. In between each diagram is an explanation of what was done, how it affects the mathematics, and what will be done in the next diagram. Color could play a role here, too. Remember that every diagram has a figure number and a descriptive caption.

Andrew created the series of diagrams that appears below. The figure numbers, captions, and explanatory sentences in between each diagram were omitted so you could focus on the progression of increasingly complex diagrams. As you look at the following graduated series of diagrams, focus on the last diagram. Imagine if that was the first, and only diagram the reader had to look at. Can you see how much the graduated series of diagrams helps the reader follow the development of the material?




5
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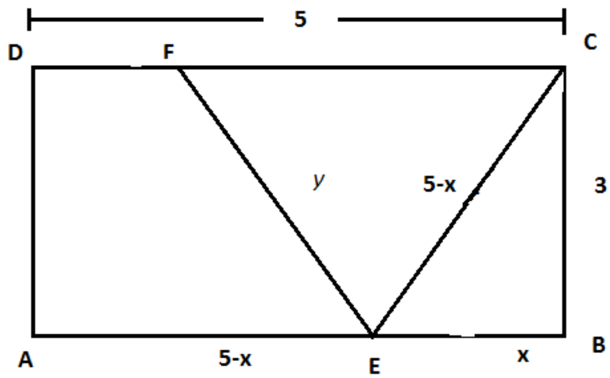
x
B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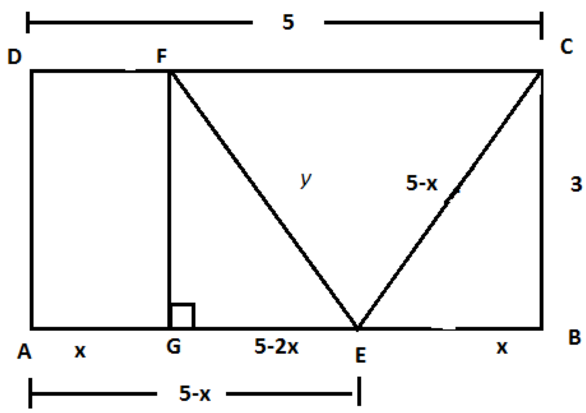
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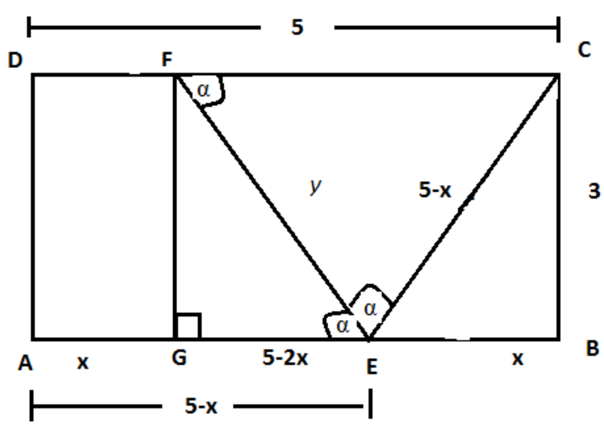
5
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3
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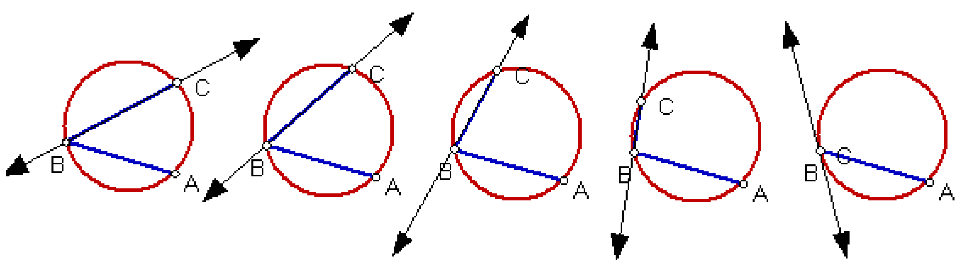
α
A
x
G
5-2x
E
x
B
5-x

Notice that as the diagrams progress, one or two line segments and labeled points, segments and angles are added, and/or other changes were made. Explanatory sentences must go in between each diagram, and each diagram must have a descriptive caption. This way, the "busy" diagram unfurls one step at a time, and the reader knows the order of the development.

Tip 30. Use graduated diagrams to simulate animation.

Animation helps illustrate mathematical ideas that express change through various parameters. You can't have an animation on paper, but you can have the next best thing; you can use dynamic logic in a series of diagrams to simulate motion. Each picture of the series is like a frame of a movie, but each picture shouldn't be a separate figure. A series of pictures next to each other is what helps the reader understand the mathematics.

Let's say you just completed a lesson showing that the degree measure of an inscribed angle is equal to one-half of its intercepted arc. Now you want to show a reader that the degree measure of an angle formed by a tangent and a chord is equal to one–half of its intercepted arc. Imagine an animation of an inscribed angle actually turning into an angle formed by a tangent and a chord:



As C gets closer to B, different inscribed angles are formed. However, each inscribed angle is equal to one-half the measure of its intercepted arc. When C is really close to B, angle ABC is still inscribed, and still measured by one-half its intercepted arc. When C reaches B, the segment BC "becomes" a tangent line, and angle ABC is an angle formed by a tangent and a chord.

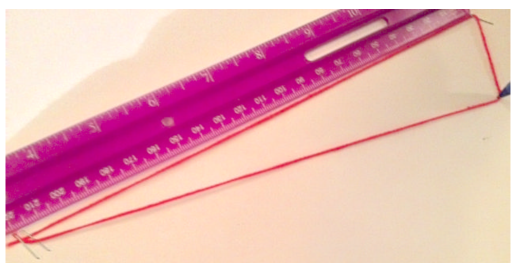
Based on the sequence of BC segments in the figure, it makes sense to conjecture that the measure of an angle formed by a tangent and a chord is measured by one–half of its intercepted arc.

Notice the purpose here is different than the previous use of graduated diagrams to build intricate diagrams. The diagrams in this development were not intricate; they required graduated diagrams for a different reason. The author needs to describe what the series of diagrams is trying to show. Keep in mind that if your diagrams are done by hand with a compass and straightedge, you can scan them right into your document, or draw them by hand in your document.

Tip 31. Include photos of manipulatives.

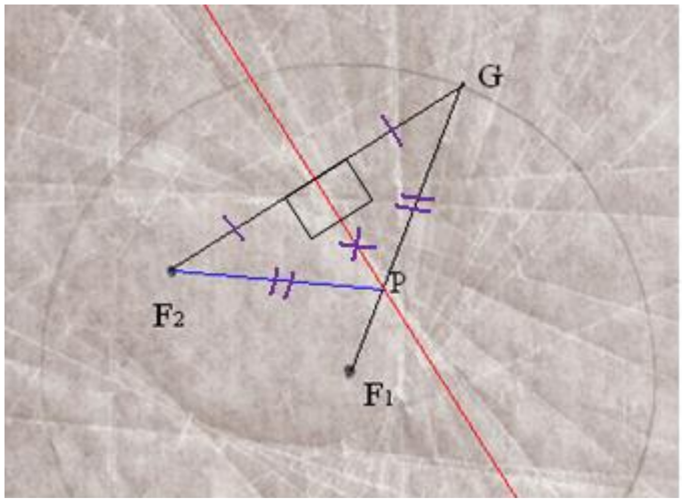
A manipulative is any physical item you use in you mathematics education. If you have a manipulative that you cannot include in the written paper, you can describe how it works by taking a series of pictures of it in action. You can write captions for each picture and relate the pictures to the mathematics. This is another form of graduated information, but it uses photographs taken with a camera.

If you have a digital camera or cell phone, or tablet, the pictures can be pasted right up into your paper. Aidan was doing research on bending a 48cm wire to form a right triangle. She was testing some theories she had using string and a ruler, and she took pictures of her findings.



Katerina was working on a problem involving the earth's circumference, so she used the big circle on the school gym floor to test some of the claims made in the article she was reading. She needed the help of some friends to properly place string around the large circle, as shown below.

Meghan used small sheets of wax paper (called patty paper) to do a paper folding exercise. After following directions to create many creases in the patty paper, she noticed an ellipse starting to form, as shown below. She took a photograph of the patty paper with all of the creases, and pasted the photo into a drawing program, which she used to explain some mathematical concepts.



Be sure to include the figure numbers and captions for each photograph.

Tip 32. If you write or use a computer program, explain how it works.

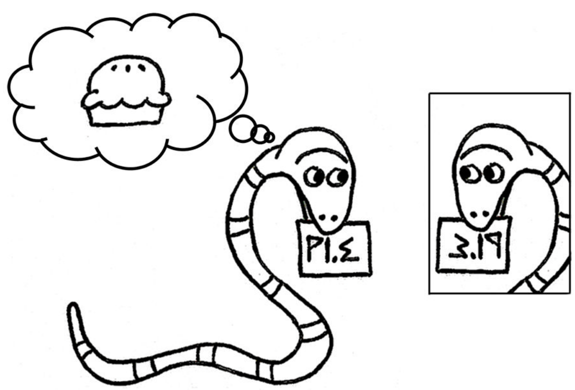
If your work requires the use of a computer program that you need to write or have written for a computer or for a programmable calculator, you need to list the entire program and explain each step of what the program does. You may have written the program, or copied it from some source. Maybe a computer teacher or a computer student helped you write it. If you are using it as part of your work, you and the reader need to understand the function of each line of the program.

Look at Aidan's program, written on his graphing calculator. Notice how the function of each line is explained. Also notice how he highlighted alternate lines of his programs, almost as if the program and explanations were written on blue and white computer paper. Aidan also enlarged the lines of the program and shrunk the font for the explanations so the explanations would fit neatly next to the line in the program they were referring to. Try to keep your computer program on one succinct page, if possible.

:0→R	In our program, R will represent the number of points under the curve out of the 10,000 random points. In this step, we are making R equal to 0 before the program starts.
:For(A,1,10000,1)	In this next step, we are setting the program to loop. "A" represents the variable. The next two numbers, 1 and 10,000, mean that we want 10,000 loops of this program. The last number, 1, means that the interval of the numbers between 1 and 10,000 is 1.
:(3+rand5)→X	Here, we are setting a random number to be X. Since rand usually starts at 0 and we are adding 3, the lowest the random number can is 3, and since the next number is 5, we will add 5 to 3, getting 8, meaning that 8 is the largest number X can be. As a result, X can be between 3 and 8, inclusive. This is because the rectangle goes from 3 to 8 on the x-axis.
:(rand51)→Y	In this step, we are setting Y to be equal to a random number between 0 and 51. This is because the rectangle's lowest height is 0 and maximum height is 51.
:If $Y \le X^2-2X+3$	Here, we begin to set up the program by telling the calculator that it will do something if the point is on or below the curve $Y \le X^2-2X+3$.
:Then	In this step, we set the calculator up to do something if the point is below or on the curve $Y \le X^2-2X+3$.
:R+1→R	Here we program the calculator to increase R, which started at 0, by 1 each time that a point is on or below the curve. This way we can find out how many points out of the 10,000 were at or under the curve.
:Else	With "Else", we are telling the calculator what to do if the point is not at or below the curve.
:End	Now we are programming the calculator to end the "If-Then-Else" subroutine of the program.
:End	With the second "End" we are telling the calculator to finish and test the next point.
:Disp R	With this line, we are programming the calculator to display the value of R after all of the trials have taken place. We will then use this number to figure out the area under the curve.

Tip 33. Revise your paper frequently.

When you begin to practice your mathematics writing, concentrate on using the writing tips in *Write On! Math*. Eventually they will become second nature; you won't have to concentrate on the techniques. As you create pages, read them over for errors. Make corrections before the next draft. Read these same pages again when you prepare subsequent drafts. This is a crucial part of the revision process. It is like looking in a mirror to find things that can be improved.



Writing is like combing your hair, or fine-sanding a piece of wood. The more you do it, the smoother it gets. As you do it, you look for the rough spots, and pay extra attention to them. Then you redo the job again, and most likely, you'll find other spots that need attention. The more times you proofread and revise a written piece, the better it gets. Every time you generate a few pages, check them over. When the final document is done in draft form, pay attention to the general flow.

There is a difference between proofreading and editing. Proofreading is language–specific. When you proofread, you check for grammar, spelling, paragraphing, sentence structure, correct usage of words, etc. Computers help with this, giving you indications of spelling errors and even some grammatical errors.

Editing is subject-specific. When you edit a manuscript, you are looking at the content of the manuscript. Is it correct? Is it presented in a logical order? Would an extra table or diagram help the reader? Should we discuss a particular topic in greater detail? Should we give another example of something? These are all questions an editor faces.

An editor must have background knowledge of the subject matter of the manuscript; a math editor must know mathematics. Technically, someone who does not know mathematics could proofread a mathematics paper, but it is preferable to have it proofread by someone who knows both mathematics and writing techniques. So, in the final analysis, proofreading and editing are really intertwined, and go under the broad heading of revision.

Proofread and edit as you write. Each time you create a few pages or paragraphs, proofread and edit them. Each time you read the new pages, read the old pages, too. It's amazing, but you will find that you will still be fine–tuning the first pages, even after having read them dozens of times.

Tip 34. Seek writing help if you need it.

If you have questions about general English language grammar or usage not covered in these writing tips, you can consult an English teacher or a writing manual. There are many online sites that can help you with grammatical questions.

Tip 35. Get a naïve proofreader.

What is a naïve proofreader? A naïve proofreader is a reader who is not familiar with the intent of the author. When you write and revise, you will always be switching roles. You will write material that you understand well, but you will try to proofread it as someone unfamiliar with the material. You need to empathize with someone who does not have proior knowledge about your work. This is difficult to do, and you will get better at is as you practice it.

Also consider getting a naive proofreader. Do not discuss the paper with the naïve proofreader. The naïve proofreader reads what you wrote and becomes a terrific barometer of whether or not the words covey the intended message. Have the naïve proofreader read your paper and mark up any parts they don't understand. Then you can revise that part.

Who can you get to be a naive proofreader? A good naïve proofreader would be another student who has the necessary background to understand your mathematics. Usually it is a student in your grade level, or even your mathematics class. It could be a friend.

Go back and read the example about the voters on Long Island on page 000. Think about the people who did the proofreading of the proposal before it was put on a ballot. They were probably the people who were working on the proposal all along. Imagine how a naïve proofreader could have helped improve the wording of the proposal. The person who proofread that proposal knew the intent of the proposal and could not imagine it being interpreted any other way.

Can you appreciate the value of a naïve proofreader?

Tip 36. Use the writing tips checklist as you write.

Below is a list of each of the writing tips contained in this section of *Write On! Math*. Use this checklist as you write to make sure you are following each tip.

Tip Number	Tip
1	Create an outline.
2	Save your work frequently.
3	Use standard, academic-looking fonts.
4	Number the pages and use a footer.
5	Double-space your drafts.
6	Use an equation editor to create mathematical expressions.
7	Format your mathematics like a technical writer.
8	You can “wrap” your left justified reasons and explanations.
9	Do not “wrap” your centered algebraic expressions.
10	Number equations that will be referred to later.
11	All figures must be numbered, with a descriptive, one-sentence caption.
12	There are several ways to display graphs.
13	Use the words “We”, “They”, and “Let’s.”
14	Don’t use judgmental phrases.
15	Separate your manuscript Into different sections.
16	Use segues at the beginning and end of each subsection.
17	Create a descriptive title for your work.
18	Make an academic–looking, sophisticated, informative cover page.
19	Write an abstract on the page after your cover page.
20	Label geometric diagrams properly
21	Don’t start a sentence with a variable, numeral or symbol.
22	Use an Appendix for excessive amounts of data.
23	Use correct mathematical terminology.
24	Carefully note how you use pronouns.
25	Rewrite entire equations as you manipulate them.
26	A table must be centered on the page.
27	Develop complicated tables gradually, one step at a time.
28	Use color creatively, but with discretion.
29	Use a graduated series of diagrams to develop an intricate diagram.
30	Use graduated diagrams to simulate animation.
31	Include photos of manipulatives.
32	If you write or use a computer program, explain how it works.
33	Revise your paper frequently.
34	Seek writing help if you need it.
35	Get a naïve proofreader.
36	Use the writing tips checklist as you write.

Eventually, you might not need the checklist—you will be adept at technical writing!

Taking the Write Turn

The purpose of *Write On! Math* is to systematically give you a method to improve your mathematics writing. With this Guide and the examples of student written mathematics, you are equipped to write professional looking mathematics and technical material. If you play a sport or a musical instrument, you understand the value of practice. You know that the only way to perform better is to get out on the field or in the field. The more you practice, the better you will perform.

You now have the tools to become a talented technical writer. Use them. This ability will reap dividends in parts of your writing across all disciplines. Your science lab write-ups, and even papers in the humanities and social sciences, will benefit by using these tips wherever technical material needs to be explained.

Use these materials as a reference when you write and when you proofread. Use the student work frequently to see examples of the writing tips in action. You'll be proud to hand in your work when it is done so professionally.

Some people seem to have a natural gift for mathematics. Others have to work very hard to achieve success. No matter where you fall in this spectrum, you will need to express yourself using the written word. By now you understand how writing can help you understand material better.

A typical way to become a better writer is to read. However, reading mathematics is quite different than reading the sports pages or a novel. It can take minutes or hours to read and re-read just one paragraph to get the gist of what it is saying. Reading mathematics is not easy reading. An alternate way to becoming a better writer is to annotate your notes regularly, and annotate your annotated sentences to make them smoother and clearer. Organize and revise your annotations. This practice will make you a better technical writer.

So get in the habit of recopying your notes; it is a very effective way to learn mathematics too. After recopying your notes, write out questions you have. See your teacher and ask these questions. Add the answers to your notes. Sift out all of the things you don't understand as you learn them; don't wait until the night before a test. You'll stay current and you'll be primed to follow all class presentations. It is worth the investment of time. It will improve your grade and your work and study habits.
This is the "write" way to start.

> *"A good notation has a subtlety and suggestiveness which at times makes it almost seem like a live teacher."* — Bertrand Russell (1872–1970)

Become your own "live teacher"!

Chapter 7

Conjectures, Theorems, and Proofs

Before you could speak, you were recognizing patterns and making generalizations. You could predict events of the day based on your experiences with the sequence of events in previous days. You even made conjectures before you could crawl! You hypothesized that it was near lunchtime, bedtime, and so on. You hypothesized that it was bath time when you heard the water running in the tub at a certain time of day. As a young adult, you make more sophisticated conjectures about everything from politics to fashion, and you have the ability to verbalize your conjectures. Did you ever realize how often you make conjectures in a day?

Suppose you are to meet your friend Ryan at a certain restaurant at 3:30 P.M. If by 3:55 he hasn't shown up, how many conjectures run through your mind? Your questioning and problem-solving skills are working so naturally you may not even realize they are operating!

- Did Ryan forget?
- Is this the wrong meeting place?
- What activity could have held up Ryan?
- Where should I look for him?
- Where was his last class?
- Was he taking the bus or walking today?
- Could my watch be wrong?

Making conjectures and testing are an integral part of mathematics research. Proof is the language by which much of mathematics is communicated. The ability to prove mathematical claims sets mathematics apart from other disciplines. With proofs, mathematicians can explain or demonstrate *why* their findings must be true. When someone makes a mathematical discovery and proves it, the discovery becomes incontestable.

The Role of Proof in Historical, Scientific, and Mathematical Research

Most students carry out their first research in a history class, usually by writing an expository paper on some aspect of history. Keep in mind that with historical research the reader can never be 100% certain that an argument used to back a theory is correct. History papers can involve an original theory (the Kennedy assassination was planned by . . .), but a proof for such a theory is very hard to uncover because you are relying on the interpretation, selection, and validity of previously written sources. If, through convincing historical evidence, a theory based on historical research is accepted as "true," it would probably be difficult to say that it was *definitely* true, beyond any doubt. Even video footage often does not conclusively prove anything. It would be hard to place a percentage value on whether a given theory is true. Are you 99% sure? 60% sure? Is it reasonable to even try to apply a number to a qualitative argument? Since historical research deals with events that have already happened, the probability of any theory being true is either 0 or 1. If the theory is true, the probability of it being true is 1. If the theory is false, the probability of it being true is 0. Unfortunately, the existing evidence does not guarantee you came to a correct conclusion.

The results of many scientific research experiments have a probability of being true. In certain scientific experiments, we can compute a probability that the results are accurate and base our conclusions on the rigor of the experiment and the value of the probability. You will learn in statistics that many scientific tests involve taking a sample that will represent an entire population. Scientists try to take random samples to avoid bias in selecting the sample. With random sampling, some samples are better representatives of the population than others. But some are actually very poor representatives of the population. Information obtained from these unrepresentative samples can be misleading. Mathematicians can compute the probability that a sample will be a good representation of the population, but this probability will never be 100%. If an experiment has a 99% chance of being correct, will scientists believe it is accurate? If the experimental design is solid, they will. Policies may be formed and further research will take place based on the results. If a scientist makes a hypothesis that is not supported by the results of an experiment, the result is still important. It may indicate that the hypothesis is *not* true.

While historians and scientists must explain why they believe their theories are probably true, the proof of a mathematical conjecture leaves no doubt of the conjecture's correctness as long as its proof is correct. A conjecture, when proved, becomes a theorem. When disproved with an argument or a counterexample, a conjecture is false. Until a proof (or disproof) is found, a math conjecture is much like a theory in a history research paper—we don't know if it is true. Even if we have a strong conviction that the conjecture is true, we can't explain why without a proof.

Unlike historical research, math research always involves the chance that a mathematician will discover a proof that we are 100% certain is correct. It may take centuries to finally prove something, and proof attempts—both successful and unsuccessful—can lead to whole new discoveries and the development of entire branches of study. Recall from Chapter 1 that the most famous contemporary breakthrough in mathematics was the proof of what has been called "Fermat's Last Theorem." That theorem had no known proof, and remained a conjecture, for over 300 years! Over the years, unsuccessful attempts to prove the theorem led to the discovery of commutative ring theory and a wealth of other mathematical findings.

Proofs form the cornerstone of much mathematics research. Before you can create proofs on your own, however, you need experience in reading and explaining the completed proofs of others.

Theorems and Proofs as Part of Your Research

Most of the articles you read will contain proofs of the claims that are made. Before you become actively engaged in reading mathematics, you need to become familiar with the types of proofs you will encounter in your readings. Perhaps you have some experience in writing proofs from units in geometry or logic studied in your math courses. At this point in your mathematics education, your experience with proofs might be limited. A good way to acquire knowledge about proofs is to read through proofs in journal articles or in a math textbook. See if you can explain the reasoning behind each line in a given proof. Reading and explaining "between the lines" will help orient you to the process of proofs. It can be especially helpful to do this with another student or in a small group. Exposure to finished proofs is a building block in learning to create original proofs of your own. In your research project, you can handle theorems and proofs on several different levels.

- You can explain the reasons for each step of the proofs given in the articles you cite. Often, the reasons are omitted because the target audience of the journal is mathematics instructors, who have experience with proofs.
- You can add steps in between the lines of the proofs given in an article. It is often assumed that the readership of the article can figure out the steps that are missing. In your research project, you can present and explain these missing steps. We call this "reading between the lines" in mathematics.
- You can test the claims made in an article. If a theorem is presented, whether or not it is proved, try a few examples to verify the claim. Include these original examples as illustrations of the theorem in your research project.
- You might make a conjecture based on information found during the course of your research. Illustrate your conjecture with original examples. At times, you might disprove your own conjecture with one of your examples. The conjecture and counterexample can still be included in your research project.

- You might actually try to prove a conjecture you make. If so, you may need some guidance from your instructor in planning such proofs.

Tailor your treatment of proofs to your research. This chapter offers an overview of proofs similar to the overview of problem solving in Chapter 2. Just as Chapter 2 couldn't possibly provide all the experience you need in problem solving, the aim of this chapter is merely to touch on a few examples of proofs, primarily to help you in your reading. You'll need to draw on other math course experience to become an accomplished writer of proofs. As you read the chapter, concentrate on the direct proofs taken from actual student papers. Use this chapter as a reference as you encounter proofs in your article. Make sure you give the readers of your research thorough explanations of the proofs related to your article, whether they were written by you or by the article's author.

Making Mathematical Conjectures

Many of the conjectures you make in daily life require estimating and having number sense. Hypotheses about money, time, temperature, and so on are somewhat mathematical because they involve numbers. In your research, you will usually make conjectures based on purely mathematical situations. Let's take a look at some mathematical conjectures.

Find the next two terms of this sequence:

$$2, 4, 6, \_\_\_\_, \_\_\_\_, \ldots$$

For most people, the numbers 8 and 10 seem logical. Are they the only choices?

- If the rule is "Add the previous two numbers to find each term," then the next two numbers are 10 and 16.

- If the rule is "Start with 2 and 4, and add all previous numbers to find each term," then the next two numbers are 12 and 24.

- If the rule is "Start with 2 and 4, multiply the two previous numbers, and then subtract 2," then the next two numbers are 22 and 130.

Which answer is correct? Based on the information given, all are equally logical. There are probably dozens of other possibilities. But how many people are mathematically open-minded enough to use their imaginations and insight to search beyond the common pattern 2, 4, 6, 8, 10? Your ability to study patterns for their obvious and not-so-obvious properties is crucial to your ability to conjecture. Examine the following set of patterns, determine in which sequence the number 15 would fit, and explain the rule used to generate that pattern.

$$1, 4, 7, 11, 14, \ldots$$

$$0, 3, 6, 8, 9, \ldots$$

$$2, 5, 10, 12, 13, \ldots$$

Be open-minded! If you don't want to see the solution, read no further. If you are working on arithmetic algorithms to find a logical connective between each of these terms, you are probably going to get frustrated. The number 15 would be the next term in the last sequence. Why? The first sequence is whole numbers written with straight-line segments only. The second sequence is whole numbers with curved digits only. The third sequence is whole numbers that are written with both curved and straight lines. The moral? Use your "mathimagination" in searching for patterns. As your mathematics background, intuition, and insight develop, you will learn to make conjectures based on patterns you observe. Some conjectures may turn out to be true; others, false. When you make a conjecture, it must be reasonable based on the information given. That is, the conjecture must hold for the cases you've seen. In Chapter 1, you encountered several patterns and conjectures as a preview. Now you are ready to concentrate on forming simple and then more complex mathematical conjectures.

Study each of the following examples and try to form conjectures based on the information given.

- The list of perfect squares is infinite:

  $$1, 4, 9, 16, 25, 36, \ldots$$

  What conjectures can you make about patterns in this sequence?

- Inspect this list of Pythagorean triples:

  $$3, 4, 5$$
  $$5, 12, 13$$
  $$7, 24, 25$$
  $$9, 40, 41$$
  $$11, 60, 61$$
  $$13, 84, 85$$

  What conjectures can you make about the patterns in this sequence?

- The set of prime numbers is infinite:

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, \ldots\}$$

What conjectures can you make about the prime numbers?

- The Fibonacci sequence is a recursive sequence (see Chapter 1):

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$$

Try making some conjectures about the Fibonacci sequence.

After considering each of the four sequences above independently, can you think of some conjectures based on combinations of the above concepts? Are there Pythagorean triples composed of all prime numbers? Of only composite numbers? Are any perfect squares prime numbers? Are any Fibonacci numbers perfect squares? Is there a number that is prime, a Fibonacci number, and the length of the hypotenuse in a Pythagorean triple? As you ask yourself probing questions based on the sequences and how they are formed, you might decide to test some more cases. Formulate a conjecture based on your question and its truth for the limited number of trials you made. If you were doing research in the eighth grade on geometry, do you think you would have formulated some of these conjectures by examining, and "playing with," many different figures?

- The base angles of an isosceles triangle cannot be right angles.
- The base angles of an isosceles triangle are congruent.
- The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
- The area of a square with side x is greater than the area of a circle with diameter x.
- If a parallelogram is not a rectangle, its four vertices cannot lie on one circle.

In the next section, we will look at some conjectures made by students and describe several types of proofs used in mathematics.

Examples of Mathematical Proofs

Before examining different mathematical conjectures and their proofs, we offer a warning necessary in light of the tremendous technological advances being made today. As computers and graphing calculators become as common as paper and pencil, we are gaining a terrific tool for finding patterns and making conjectures. Machines can test many cases, allowing us to present quite a convincing argument for a conjecture that holds true after extensive testing. Keep in mind that these long lists of numbers that result from extensive testing do not constitute a proof. For example, a computer printout of the first 125,000 twin-prime pairs (see Chapter 1) does not mean there are infinitely many twin-prime pairs. Such lists certainly can display enough data to make our effort to *find* a proof a reasonable undertaking, but they are not proofs in and of themselves. We will discuss direct proofs, indirect proofs, and proofs that use the Principle of Mathematical Induction.

Direct Proofs

There are times when conjectures can be proved by testing every possible case. This is not common, but it can occur, and it is one form of direct proof. In fact, it's one type of proof that a computer is capable of performing. Let's look at two examples:

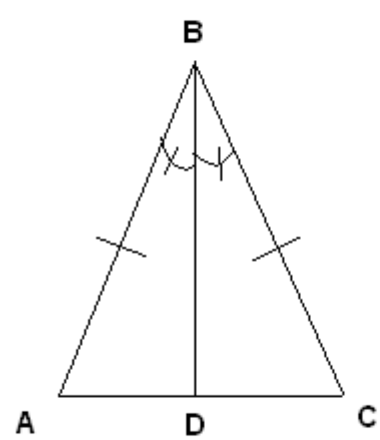
• Julianne was reading an article from the February 1991 issue of *Mathematics and Informatics Quarterly,* titled "What Is the Use of the Last Digit?" The article was written by Lyubomir Lyubenov. The first part of Julianne's research required her to prove that, for all whole numbers x, the units digit of x^5 was the same as the units' digit of x. Because the units' digit is the only one in question here, Julianne raised each of the digits 0 – 9 to the fifth power and found that the units' digit of x^5 is indeed equal to the units' digit of x. This result can be obtained rather quickly either by hand or by using a scientific calculator, and it constitutes the proof. Can you use this fact to prove that $x^5 - x$ must be divisible by 10?

• Ben was doing research on the Platonic solids. There are five Platonic solids: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. Ben counted the number of faces (F), edges (E), and vertices (V) on each Platonic solid. After working with just three of the solids, he formulated this conjecture:

$$F + V - E = 2$$

He tested his conjecture by counting the parts of the other two solids and found that his theory was correct. His proof was complete because he tested the data for all possible cases. (This theorem is known as Euler's Formula, but Ben had never heard of it, so this result was truly "unknown" to him). Later he was able to find alternate proofs of his conjecture. Ben then wondered why there were only five Platonic solids and set out to prove that fact. He also planned to extend his work into other types of solids.

Perhaps you have used direct proofs in geometry. Let's review the proof of the following theorem: If two sides of a triangle are congruent, the angles opposite those sides are congruent. The figure shows triangle *ABC* with angle bisector *BD*. Sides *AB* and *BC* are congruent.



The plan is to prove triangle *ABD* congruent to triangle *CBD*, using a **paragraph proof**.

These two triangles already have two pairs of congruent sides given.

$$\overline{AB} \cong \overline{BC}.$$

BD is common to both triangles, so, by the reflexive property,

$$\overline{BD} \cong \overline{BD}.$$

Since *BD* is an angle bisector,

$$< ABD \cong < CBD.$$

Triangle *ABD* has two sides and an included angle congruent to two sides and an included angle of triangle *CBD*, respectively, therefore,

$$\Delta ABD \cong \Delta CBD.$$

Since corresponding parts of congruent triangles are congruent,

$$< A \cong < C.$$

We have used deductive reasoning and theorems in geometry to prove that base angles of isosceles triangles are congruent.

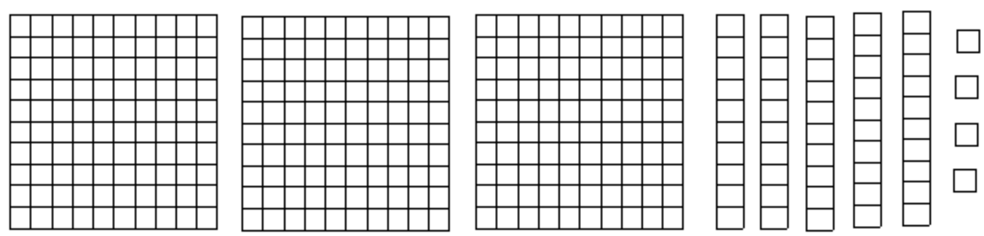
You may have experience with **two-column proofs** like this written in a two-column table as shown below.

Statements	**Reasons**
1. $\overline{BD}$ bisects $< ABC$	1. Given
2. $< ABD \cong < CBD$	2. Definition of angle bisector
3. $\overline{AB} \cong \overline{CB}$	3. Given
4. $\overline{BD} \cong \overline{BD}$	4. Reflexive property of congruence
5. $\Delta ABD \cong \Delta CBD$	5. SAS ≅ SAS
6. $< A \cong < C$	6. Corresponding parts of congruent triangles are congruent.

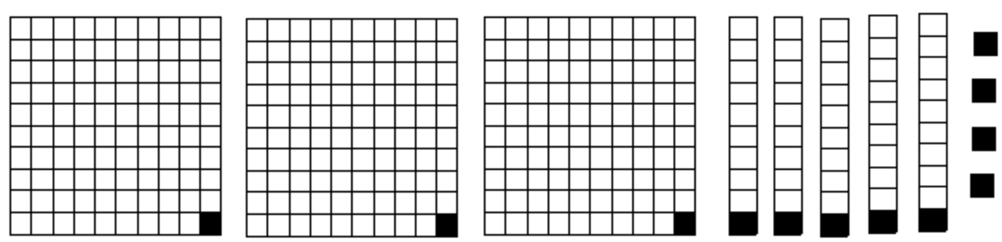
These paragraph and two-column formats provide a useful shorthand for writing proofs. Paragraph proofs, however, can offer commentary and greater detail. In the vast majority of math research, you will find proofs written in paragraph form.

Many direct proofs involve algebra. Variables are used because there are many, often infinitely many, cases to test. You might remember the test for determining if a natural number is divisible by 3. When you add all of the digits, if the sum of the digits is divisible by 3, then the number itself is divisible by 3. Let's demonstrate this with a geometric example and then prove it algebraically.

The number 354 is divisible by 3 because 3 + 5 + 4 = 12, and 12 is divisible by 3. Let's look at the number 354 represented on grid paper using three groups of 100, five groups of 10, and four 1s.



Shade in the four single squares. Then shade in one small square in each of the 10 x 10 squares and in each of the 10 x 1 squares.



You now have 3 + 5 + 4 = 12 shaded squares. The number of unshaded squares is divisible by 3 because it is the sum of three groups of 99 and four groups of 9. Therefore, the quantity 354 is divisible by 3 if the number of single shaded squares is divisible by 3. The number of single shaded squares is the sum of the digits, 12, and that is divisible by 3, so 354 is divisible by 3. Try this with another number!

This type of geometric demonstration would become cumbersome with numbers that have more than three digits. Let's try to prove this algebraically for a four-digit number, *ABCD*. Recall that *ABCD* can be represented in expanded form as

$$1000A + 100B + 10C + D$$

Separate the numerical coefficients so the second addend is 1.

$$(999 + 1)A + (99 + 1)B + (9 + 1)C + D$$

Distribute:

$$999A + A + 99B + B + 9\text{C} + C + D$$

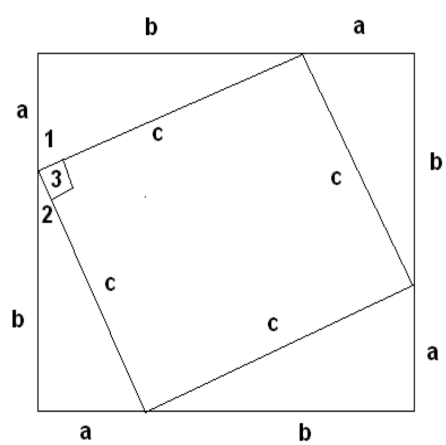
Commute:

$$999A + 99B + 9C + A + B + C + D$$

Notice that $999A + 99B + 99C$ is divisible by 3 because it is divisible by 9. The entire number *ABCD* is divisible by 3 if $A + B + C + D$ is divisible by 3, and this is the sum of the digits!

In Chapter 2, we discussed criteria for picking a topic. For many students, the Pythagorean theorem is an appropriate introductory research topic. Let's look at a direct proof of the Pythagorean theorem itself, as well as some direct proofs from students' papers on Pythagorean triples.

Given a square, mark off equal sections of length a on each side of the square, as shown on the next page. The remaining segments on each side have length b. The length of each side of the square can be represented by $a + b$. The congruent hypotenuses have length c.



The quadrilateral (rhombus) inscribed in the square is also a square. The four sides are equal because they are all essentially hypotenuses of congruent right triangles. Angles 1 and 2 are complementary, since the acute angles of any right triangle are complementary. Since angles 1, 2, and 3 form a straight angle, angle 3 must be a right angle and the rhombus is actually a square. The area of the larger square is

$$(a + b)^2.$$

Squaring this binomial yields

$$a^2 + 2ab + b^2.$$

The area of each of the four triangles is

$$(1/2)ab.$$

The total area of the four triangles is

$$4(1/2)\,ab = 2ab.$$

The area of the smaller square, c^2, can be expressed as the difference between the area of the large square and the total area of the four triangles.

$$a^2 + 2ab + b^2 = 2ab + c^2.$$

Subtract 2ab from both sides.

$$a^2 + b^2 = c^2.$$

Alexis did research on Pythagorean triples. Part of her work used the famous Euclidean method for generating Pythagorean triples. The Euclidean method involves formulas that require the user to choose two positive integers, p and q, where $p > q$, and substitute them to find the sides a, b, and c. (The derivation of these formulas could be part of a paper incorporating them.) The formulas are:

$$a = p^2 - q^2$$

$$b = 2pq$$

$$c = p^2 + q^2$$

Try some values of p and q and verify that Pythagorean triples are generated. Then, *prove* that these formulas always satisfy the Pythagorean theorem. Experimentation with these formulas can lead to many interesting patterns and conjectures. Alexis used a spreadsheet to generate many sets of triples using this formula. Table 7.1 shows a portion of her spreadsheet.

Table 7.1. Pythagorean Triples Formed Using the Euclidean Method

p	*q*	*a*	*b*	*c*
2	1	3	4	5
3	2	5	12	13
4	3	7	24	25
5	4	9	40	41
6	5	11	60	61
7	6	13	84	85
8	7	15	112	113
9	8	17	144	145
10	9	19	180	181

After fifty trials on the spreadsheet, Alexis noticed that whenever p and q were consecutive integers the larger leg and the hypotenuse differed by 1. She wanted to prove this conjecture; she believed it to be true. She started with the formulas that generate b and c. Then she substituted $q + 1$ for p in both formulas.

$$b = 2pq$$

Substitute.

$$b = 2(q + 1)q$$

Distribute.

$$b = 2q^2 + q. \qquad \text{(Equation 1)}$$

Use the formula for the hypotenuse, c.

$$c = p^2 + q^2.$$

Substitute.

$$c = (q + 1)^2 + q^2$$

Square the binomial.

$$c = q^2 + 2q + 1 + q^2$$

Combine like terms.

$$c = 2q^2 + 2q + 1.$$

Compare this result to Equation 1. Notice that c is 1 more than b, so Alexis' conjecture has been proven for all cases, and it is actually a theorem. Alexis's direct proof shows that b and c differ by 1.

Sharon found another pattern using results from a table of Pythagorean triples generated using the Euclidean formulas (above) and a calculator. Looking at the triples generated using $q = 1$ and p varying from 2 through 12, Sharon noticed that the difference between the length of the short leg and the hypotenuse is always a perfect square. (See if you can verify her findings). She conjectured that this was true for all values of p greater than 3, and set out to prove her theory.

$$q = 1$$

Since $q = 1$, the short leg b is

$$b = 2pq = 2p.$$

Since $q = 1$, the hypotenuse c is

$$c = p^2 + q^2 = p^2 + 1.$$

The difference between the short leg and the hypotenuse is

$$c - b = p^2 + 1 - 2p.$$

Commute.

$$c - b = p^2 - 2p + 1.$$

Factor.

$$c - b = (p - 1)^2.$$

Since p is an integer, $p - 1$ is an integer and $(p - 1)^2$ is a perfect square.

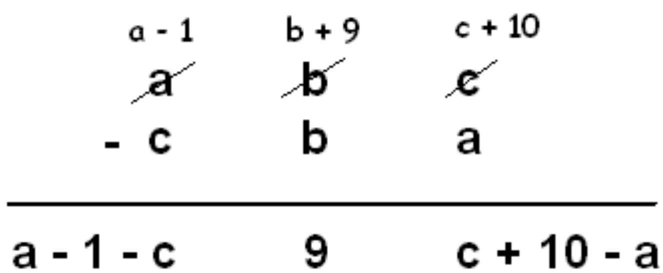
Did you notice that $b = 2pq$ represented the shorter leg in Sharon's proof and the longer leg in Alexis's case? Can you explain why? Do you see how new discoveries lead to new questions and, ultimately, to new conjectures as we attempt to answer these questions? Simran made a conjecture about primitive Pythagorean triples. **Primitive triples** are sets of three Pythagorean numbers that have the number 1 as their greatest common factor. For example, 3, 4, 5 is a primitive triple, but 6, 8, 10 is not. After generating many primitive Pythagorean triples, Simran noticed that the length of the hypotenuse was always prime or a multiple of 5. He made several attempts and "chipped away" at part of the proof that this was true for all such triples, but he could not complete the proof before the end of the course. This became an excellent problem for future researchers to work on.

Laura was doing research on special properties of subtraction in different bases. She did many numerical examples to help her discover patterns. Some of her examples are shown here:

971 - 179 = 792
865 - 568 = 297
421 - 124 = 297
521 - 125 = 396

This part of her research inspired her to conjecture that, if a three-digit number is written in ascending order of its digits and a new number is created using the same digits in descending order, the tens digit of their difference must be 9. Here is Laura's direct proof:

Given the number *abc* where $a > b > c$. From *abc* subtract *cba*. This will require regrouping since $a > c$.



The patterns in the numerical examples led Laura to her conjecture. The proof showed that her conjecture was actually a theorem.

You've now seen several examples of direct proofs. Sometimes other types of proofs are more appropriate. Experience will help you decide what type of proof to use and how to execute your proofs. Let's take a look at another type of proof you might decide to use in your research—indirect proofs.

Indirect Proofs

Did you ever answer a question on a multiple-choice test by eliminating all the answers except one and then using that answer as your choice? If so, you did not *directly* answer the question. You found the answer *indirectly,* by showing that all the other possibilities could not be correct. This method is commonly used in mathematical proofs. Such proofs are called **indirect proofs.** If the other possibilities lead to statements we know are not true (creating a **contradiction),** then those possibilities do not hold. We start an indirect proof by making an assumption. This assumption is the only part of our proof that is in doubt; the rest of the proof follows all laws of mathematics. If we obtain a contradiction, then the only step that could be false is the original assumption. We can then switch our belief from the assumption to the other possibility. This process of proving theorems indirectly is sometimes called **reductio ad absurdum.** Let's take a look at two indirect mathematical proofs.

The set of real numbers is the union of the set of rational numbers and the set of irrational numbers. Rational numbers can be expressed as the quotient of two integers; irrational numbers cannot. Prove that $\sqrt{2}$ is an irrational number.

There are two possibilities:

1. $\sqrt{2}$ is rational.

2. $\sqrt{2}$ is irrational.

Rather than proving directly that $\sqrt{2}$ is irrational, we will show that it can't be rational. Assume that $\sqrt{2}$ is rational. Then $\sqrt{2}$ can be expressed as the quotient of two integers, *a* and *b,* where *a*/*b* is in simplest form:

$$\sqrt{2} = \frac{a}{b}.$$

Multiply both sides by b.

$$\sqrt{2}\ b = a.$$

Square both sides.

$$2b^2 = a^2.$$

The left side of the equation is even, since it has a factor of 2. Therefore, the right side of the equation, a^2, is even. If a^2 is even, then *a* must have a factor of 2 and thus is also even. If *a* is even, it can be represented as 2*n*, where *n* is an integer. Then

$$2b^2 = a^2 = (2n)^2 = 4n^2.$$

Therefore,

$$2b^2 = 4n^2.$$

Divide by 2.

$$b^2 = 2n^2.$$

This tells us that b^2 is even, so as a result, b is even. Since a and b are both even, the fraction a/b is *not* in simplest form as originally assumed—it can be reduced by the factor of 2 in the numerator and denominator—contradicting the assumption that $\sqrt{2}$ can be expressed as a fraction in simplest form. Therefore, $\sqrt{2}$cannot be expressed as a fraction in simplest form and must be irrational.

The following is an indirect proof from a student's paper on Pythagorean triples.

Lauren used some mathematical notation in her proof of a conjecture about the **parity** (evenness or oddness) of numbers in Pythagorean triples. Lauren employed the symbol |, which means "divides evenly" or "is a factor of." (For example, 7 | 28 and 3 | 12.) She also used the arrow to represent "implies." Lauren conjectured that, in a primitive Pythagorean triple, only one of the numbers *x, y,* and *z* can be even. She proved this by showing that none of the other possibilities could work. Here are the four possibilities:

1. All three numbers, *x, y,* and *z,* are even.

2. Exactly two of the numbers are even.

3. Exactly one of the numbers is even.

4. None of the numbers are even.

Lauren ruled out choice 1 because if all three numbers are even they have a common factor of 2 and do not form a primitive triple.

Lauren ruled out choice 4 by showing that if all the numbers are odd there is a contradiction. Assume *x, y,* and *z* are odd.

If *x* is odd, x^2 is odd. If *y* is odd, y^2 is odd. The sum of two odd numbers, x^2 and y^2, is even.

Since $x^2 + y^2 = z^2$, the number z^2 must be even, and since *z* itself is an integer, *z* must be even. This contradicts our assumption that *z* is odd, so our assumption that all three numbers are odd is incorrect.

Lauren ruled out choice 2 using a direct proof.

Assume *x* and *y* are even. This means 2 | *x* and 2 | *y*. Therefore, for some integers *m* and *n*,

$$x = 2m \text{ and } y = 2n.$$

(These types of expressions are commonly used to represent even numbers in proofs. How could you represent odd numbers?)

Substitute into the Pythagorean theorem.

$$(2m)^2 + (2n)^2 = z^2.$$

$$4m^2 + 4n^2 = 4(m^2 + n^2) = z^2.$$

$$4(m^2 + n^2) = z^2.$$

Let's interpret this with regards to divisibility.

$$4 \mid \text{left side of equation} \rightarrow 4 \mid \text{right side} \rightarrow 4 \mid z^2 \rightarrow 2 \mid z.$$

Since $2 \mid z$, z is even. Therefore, if two numbers in a Pythagorean triple are even, the third must be even. (Other proofs that start with the assumptions x and z are even or y and z are even follow similar steps.) Choice 2 is eliminated.

Indirectly, we have shown that the only possibility is choice 3: exactly one number is even. Interestingly, the hypotenuse z can never be even in a primitive Pythagorean triple. Given that exactly one of the numbers x, y, and z can be even, try to prove indirectly that z can't be even. (Hint: If a number is odd, it can be expressed as 2n + 1, where n is an integer.)

If you want to gain more practice with indirect proofs, borrow a geometry, precalculus, or upper-level math textbook from your library or math department. Use the index to find some proofs and carefully read through them. See if you can re-create the proofs without looking at them. Also, do an Internet search on indirect proofs. You will find a ton of information! If you get stuck on proofs in your own research, you can work through them with your instructor during your consultations. It takes years to become adept at proofs in mathematics, so expect to proceed gradually. The more experience you get, the faster you will internalize the processes.

Next we will consider a sophisticated method of proving mathematical theorems usually first encountered by seniors in high school or by college students.

The Principle of Mathematical Induction

In your research, you may encounter infinite sequences of positive integers. You might make generalizations about patterns in these sequences after observing a finite part of them. You will naturally wonder whether your conjecture holds true for the entire sequence. Remember that extensive lists supporting your conjecture do not make it true. When you make a conjecture about the rule that determines a sequence, you are using

inductive reasoning. You can test your generalization by trying more cases, but you could never try all the possible cases in an infinite sequence. Sometimes you can use algebra to prove such conjectures. The **Principle of Mathematical Induction** is a process that can be used to prove conjectures about infinite sequences.

Before looking at some examples of proofs that use this principle, let's examine two common analogies to the principle of mathematical induction.

The Ladder—Picture a ladder with infinitely many rungs stretching into the heavens and outer space. If the following two conditions hold true, you can climb all the rungs:

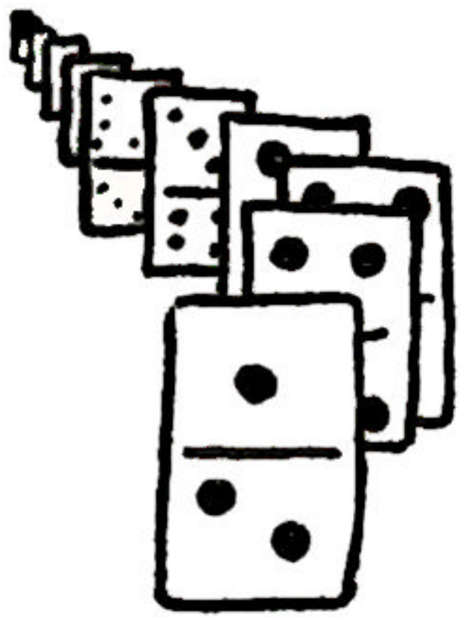
1. You are able to climb the first rung.

2. Each rung is attainable from the rung directly before it.

If *both* of these two conditions hold, then you can climb the first rung and, by "chain reaction," you can climb each rung after it, indefinitely. If only one condition holds, you cannot climb indefinitely.

Dominoes—Imagine an infinite line of dominoes. This represents an infinite sequence. If the following two conditions hold true, you can knock down all the dominoes:

- You are able to knock down the first domino.
- Each domino is able to knock down the domino immediately following it.



If these two conditions hold, then you can knock down the first domino and, by "chain reaction," each domino will knock down the domino after it, indefinitely.

These two situations are analogous to using mathematical induction to prove conjectures about infinite sequences. First, show that your conjecture holds for the first number in the sequence. Then show that if your conjecture holds for the *nth* case it also holds for the very next case, the $(n + 1)^{st}$ case. If these two conditions hold, then your conjecture is true. Here is a formal statement of the principle of mathematical induction:

Let P(n) be a statement about any positive integer *n*. If

1. The statement is true for $n = 1$ and

2. Whenever the statement is true for a positive integer ***k*,** it is also true for $k + 1$, then the statement P(n) is true for all positive integers ***n*.**

We are now ready to demonstrate the principle of mathematical induction in action. As you study each example, keep in mind the analogies of the dominoes and the ladder.

Table 7.2 shows the sum of the first *n* odd positive integers for several values of *n*. The term *xn* represents the *nth* odd positive integer. Can you make a conjecture about the sum of the first *n* odd positive integers?

Table 7.2. Sums of the First n Odd Positive Integers

n	$x_n = 2n - 1$	Sum of First n Positive Odd Integers
1	1	1
2	3	4
3	5	9
4	7	16
5	9	25
6	11	36
7	13	49
8	15	64

It is logical to make the conjecture that the sum of the first n odd positive integers is n^2. Since this is a statement about an infinite sequence of positive integers n, we should try to prove it using mathematical induction.

1. For $n = 1$: The "sum" of the first odd positive integer is 1, and $1^2 = 1$.

2. Given that the sum of the first k odd positive integers is k^2, we must show that the sum of the first $(k + 1)$ odd positive integers is $(k + 1)^2$.

The k^{th} odd integer can be represented by $2k + 1$. (Verify this with the numbers in Table 6.2.) Therefore, the $(k + 1)^{st}$ odd integer can be represented by $2(k + 1) - 1$. Also, for example, notice that you can view the sum of the first nine positive odd integers as the sum of the first eight positive odd integers added to the ninth positive odd integer. Take a look at the two bold numbers in shaded boxes in Table 7.3.

Table 7.3. The Sum of the First Nine Odd Integers Deconstructed

n	1	2	3	4	5	6	7	8	9
$2n - 1$	1	3	5	7	9	11	13	15	**17**
n^2	1	4	9	16	25	36	49	**64**	81

Using this logic, the sum of the first $(k + 1)$ odd integers is equal to the sum of the first k odd integers and the $(k + 1)^{st}$ odd integer. This is written diagrammatically below, vertically aligned with the equal sign and the addition sign.

The sum of the first $(k + 1)$ positive odd integers	=	The sum of the first k odd positive integers	+	The $(k + 1)^{st}$ positive odd integer
The sum of the first $(k + 1)$ positive odd integers	=	k^2		$+ 2(k + 1) - 1.$
The sum of the first $(k + 1)$ positive odd integers	=	k^2		$+ 2k + 1.$
The sum of the first $(k + 1)$ positive odd integers	=	$(k + 1)^2.$		

Therefore, if the sum of the first k odd integers is k^2, the sum of the first $(k + 1)$ odd integers is $(k + 1)^2$. This shows that condition 2 holds. If the formula *ever* holds, then it is guaranteed to hold for the *very next* case. This combines with condition 1 to create a "domino" effect. If the formula works for n = 1, then by condition 2 it holds for the n = 2 case. If it holds for the n = 2 case, the by condition 2 it holds for the n = 3 case. Can you see how this ingenious method of proof shows that the formula holds indefinitely? Since conditions 1 and 2 hold, the conjecture is actually a theorem, proved by the principle of mathematical induction.

If, after several test cases, the formula looked "obvious" to you, and you felt you had had no reason to prove it, notice that this formula also finds the correct sum of the first n odd integers correctly for the first million consecutive cases, but then fails.

$$n^2 + n(n-1)(n-2)(n-3)(n-4) \cdots\cdots\cdots (n - 1{,}000{,}000).$$

After checking a formula for the first 999,999 cases, would you believe it is correct? As you can see, it might not be. Therefore, as a method of proof, the principle of mathematical induction could never be replaced, even by billions of trials on a computer!

Table 7.4. Sums of Powers of 1/2

n	Sum of First n Fractions of the Form $\frac{1}{2^n}$	$1-\frac{1}{2^n}$
1	1/2	1/2
2	3/4	3/4
3	7/8	7/8
4	15/16	15/16
5	31/32	31/32
6	63/64	63/64

Table 7.4 shows values of

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots\cdots+\frac{1}{2^n}.$$

It also shows values of

$$1-\frac{1}{2^n}.$$

You might make the conjecture that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots\cdots+\frac{1}{2^n}=1-\frac{1}{2^n}.$$

We can try to prove this conjecture using mathematical induction.

1. For $n = 1$, we have a true statement:

$$\frac{1}{2^1}=1-\frac{1}{2^1}.$$

2. We now assume that the sum of the first n fractions of the form $\frac{1}{2^k}$ is $1-\frac{1}{2^k}$. Given that this holds, we must show that the sum of the first $k + 1$ fractions of this form has the sum $1-\frac{1}{2^{k+1}}$. Use the vertical alignment of each boxed explanation to better understand the line of logic and the substitutions.

$$\boxed{\text{The sum of the first } (k+1) \text{ fractions of the form } \tfrac{1}{2^n}} \;\boxed{=}\; \boxed{\text{The sum of the first k fractions of the form } \tfrac{1}{2^n}} \;\boxed{+}\; \boxed{\text{The } (k+1)^{\text{st}} \text{ fraction of the form } \tfrac{1}{2^n}}$$

$$\boxed{\text{The sum of the first } (k+1) \text{ fractions of the form } \tfrac{1}{2^n}} = 1 - \frac{1}{2^k} + \frac{1}{2^{(k+1)}}$$

$$\boxed{\text{The sum of the first } (k+1) \text{ fractions of the form } \tfrac{1}{2^n}} = \frac{2^k - 1}{2^k} + \frac{1}{2^{(k+1)}}$$

$$\boxed{\text{The sum of the first } (k+1) \text{ fractions of the form } \tfrac{1}{2^n}} = \frac{2^k - 1}{2^k} \times \frac{2}{2} + \frac{1}{2^{(k+1)}}.$$

$$= \frac{2^{k+1} - 2}{2^{k+1}} + \frac{1}{2^{(k+1)}}$$

$$= \frac{2^{k+1} - 1}{2^{k+1}}$$

$$= 1 - \frac{1}{2^{k+1}}$$

We have shown that condition 2 holds. Since conditions 1 and 2 hold, the conjecture is actually a theorem, proved by the principle of mathematical induction.

One requirement of using induction is that you must come up with a formula to test. You are not necessarily "given" what you are trying to prove—you have to come up that that on your own! Some of your problem-solving skills will come in handy here—you need to create a formula by inspecting the data in a table.

As you read each induction proof, go back to the two analogies to make sure you understand why conditions 1 and 2 cover all possible cases. If you need to apply the principle of mathematical induction in your research, first review this section. You can look online for other examples and explanations fo the Principle of Mathematical Induction.

Disproofs and Counterexamples

Sometimes your findings will lead you to make a conjecture that you eventually find out is false. To disprove a conjecture, you need find only *one* case in which it doesn't hold. This case is called a **counterexample.** If you make a conjecture and find a counterexample, do not discount the importance of your work. Report it in your research paper. If it is reasonable to make such a conjecture, then the fact that you have disproved it is essential information for readers of your paper and for future researchers of your topic.

Let's examine some famous conjectures that were disproved along with some conjectures made by students and subsequently disproved.

The famous mathematician Pierre de Fermat thought that, for all whole numbers *n,* that the following expression generates a prime number.

$$2^{2^n} + 1$$

This conjecture was based on the pattern shown in Table 7.5. These numbers came to be know as "Fermat" primes.

Table 7.5. The First Five "Fermat" Primes

n	$2^{2^n} + 1$	Is it Prime?
0	3	Yes
1	5	Yes
2	17	Yes
3	257	Yes
4	66,357	Yes

Substitute $n = 5$.

$$2^{2^5} + 1 = 4{,}294{,}967{,}296 = 641 \cdot 6{,}700{,}417.$$

The lack of computers made finding factors of larger numbers almost impossible; you can see why the pattern inherent in the first five "Fermat" numbers could lead to the false conjecture.

Ancient Chinese mathematicians conjectured that if x is an integer greater than 1 then x is a prime number if

$$x|(2^x - 2).$$

It turns out that, for $x = 341$, the conjecture does not hold.

Alexis conjectured from a list of primitive Pythagorean triples that if the short leg is an odd integer then the lengths of the longer leg and the hypotenuse are determined. In other words, there are not two primitive Pythagorean triples with the same odd-numbered short-leg length. Her conjecture was based on many trials. Eventually, Alexis found the following counterexample:

33, 56, 65 and 33, 544, 545

These triples are both primitive Pythagorean triples with the short leg equal to 33. A new question arises: Are there other Pythagorean triples with short-leg length equal to 33? Are there infinitely many more? Some experimentation will precede the formulation of a conjecture based on this question.

Robin was researching the relationship between an equilateral triangle and the three largest possible nonintersecting circles inside it. Her work led her to an inequality involving six real numbers. At one point in her work, she conjectured that for real numbers *a, b, c, d, e,* and *f,*

If $a + b + c < d + e + f$, then $a^2 + b^2 + c^2 < d^2 + e^2 + f^2$.

It seemed reasonable that the smaller sum would yield a smaller sum of squares. Try some numbers and see if you agree. Robin found a counterexample:

$a = 0.2$
$b = 0.2$
$c=3$
$d = 0.5$
$e=1$
$f=2$

Verify that Robin did, indeed, find a counterexample. Can you look at the numbers she used and determine her strategy in finding the numbers? Try to find a counterexample of your own.

Undetermined Conjectures

Any conjecture you make is either true or false. You may not know the truth value of some of your conjectures, but a truth value exists. We call such conjectures **undetermined conjectures** because their truth value is, as of now, unknown. You won't be able to prove or disprove every conjecture you make. Your conjecture will still be valid, and another researcher may attempt a proof or counterexample based on your work. That is why it is essential to offer undetermined conjectures in your research. You should always try to prove or disprove the conjectures you formulate, but when you can't, for whatever reason, do not shy away from reporting them.

Jocelyn did research on Pythagorean triples and Fibonacci numbers based on the article "Pythagoras Meets Fibonacci" in the April 1989 issue of *Mathematics Teacher*. The article mentioned some relationships between the Fibonacci numbers, the sides of the associated right triangles, and the areas of these triangles. First you need to represent four consecutive Fibonacci numbers algebraically.

$$a, b, a + b, a + 2b.$$

(Verify that, if the first two Fibonacci numbers from the set of four numbers are a and b, that the third number is $a + b$, and the fourth is $a + 2b$).

Jocelyn wondered about the relationships between the Fibonacci numbers, the associated right triangles, and the *perimeters* of the triangles. She created a spreadsheet, shown in Table 7.6.

Table 7.6. Jocelyn's Spreadsheet

Four Consecutive Fibonacci Numbers	Resulting Pythagorean Triple	Perimeter of Right Triangle
1, 1, 2, 3	3, 4, 5	12
1, 2, 3, 5	5, 12, 13	30
2, 3, 5, 8	16, 30, 34	80
3, 5, 8, 13	39, 80, 89	208
5, 8, 13, 21	105, 208, 233	546
8, 13, 21, 34	272, 546, 610	1428
13, 21, 34, 55	715, 1428, 1597	3740
21, 34, 55, 89	1869, 3740, 4181	9790

Although you may not have read the four articles Jocelyn read, you can study the table and look for patterns in it. Jocelyn finished her paper by offering several conjectures for other researchers to pursue:

- All of the perimeters are even numbers.
- Every other consecutive pair of perimeters is evenly divisible by 3.
- The perimeter is twice the product of the third and fourth Fibonacci numbers.
- The perimeter equals the sum of the squares of the third and fourth Fibonacci numbers, minus the square of the second Fibonacci number.
- If the perimeter is evenly divisible by 3, then the product of the first two Fibonacci numbers is not evenly divisible by 3.
- The perimeter on the n^{th} line is equal to the middle leg on the $(n + 1)^{st}$ line.

Jocelyn's conjectures could actually spawn an entire research paper for a reader of her work. Formulating her conjectures shows the insight and expertise that come with effort and practice.

Proving and Improving

Becoming adept at mathematical proofs takes time, patience, and effort. At several junctures in this chapter, readings were suggested to help you further your experience with mathematical proofs. If you have questions about proofs, doing an Internet search using the following key descriptors should help you:

- mathematical conjectures
- counterexamples
- deductive proofs
- deductive reasoning
- direct proofs
- indirect proofs
- inductive reasoning
- principle of mathematical induction
- reductio ad absurdum
- proof by contradiction

Print out some of the explanations so you can annotate them. Try the model problems that offer solutions. See if you can give the reasons for each step in a completed proof. Annotate the proofs. Your confidence and agility with proofs will develop with practice and exposure to proofs. Keep a reference list of online sites and books that you find to be good sources of examples of proofs. If, during your research, you have problems with proofs, it will be helpful to know exactly what source you can go to for assistance. Discuss all original proofs and attempted proofs with your instructor. The degree to which proofs play a role in your research depends on the topic you chose. You will learn much about proofs from the articles you read, and from seeing completed proofs in books and on line. Reading a proof requires patience and concentration. Chapter 8 will help you with reading mathematics.

Chapter 8

Reading and Keeping a Research Journal

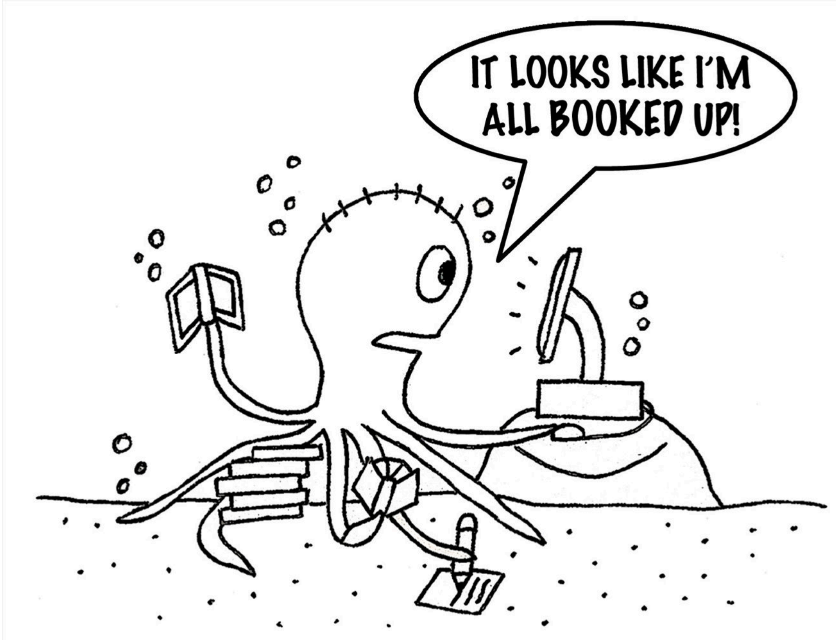
You may never have thought about the different ways you read. You may read a novel quickly. You may read the newspaper comics with a radio playing. You may simply skim parts of the newspaper's sports section looking for the results of your favorite team's game. You may skip entire sections of magazine articles as you search for a part that interests you. Road signs can be read at 55 mph, with a passing glance. You probably couldn't read an automobile insurance policy without jotting down questions. You would read a cake recipe or instructions for building a picnic table with care; an airplane repair manual would be read slowly and scrutinized even more carefully. Reading mathematics materials from your textbook or a journal article has some idiosyncrasies of its own.

Preparing to Read

When you read mathematics articles, allow ample time for each reading session. You may be able to read a newspaper article during a five-minute wait at the dentist's office, but you shouldn't allocate time for your mathematics reading in such short capsules. You'll need time to read, think, and digest information, and you'll always be writing and highlighting when you read mathematics. The degree of concentration necessary to interact with your reading precludes distractions, so don't leave the television or radio on as you read. As mentioned in Chapters 3, 4, and 5, you'll probably notice the difference between the physical layout of a mathematics article and that of a novel. The pages of the mathematics article are broken up, not filled with continuous text. In a mathematics article, portions are centered and highlighted on their own lines. Often these portions are mathematical expressions. In most cases, they are a signal to the reader to stop and digest the information. Keep this in mind when you read and when you write mathematics.

Compiling Related Resources

Each article or website you read will have a listing of references used by its author. It is helpful to make copies of some of these referenced articles in addition to your original article. Also keep a copy of all links related to your topic. You may need them sometime down the road. They will add to your research, enhance your background, and possibly lead to other avenues of research. You don't need to read them initially, and you may never need them as references, but having access to them could make your research stronger. They may offer key theorems, proofs, examples, questions, and hypotheses. Each article listed in your reference section will also have its own reference list.



Resource lists can become very lengthy; use discretion as you keep track of these extra resources. Some of the articles and links may not be helpful. Online sources sometimes shut down—something online available today may not be there in six months. Articles in popular journals may be easier to find than books that are many years old. Many article are available electronically form the publisher of the journal, although there may be a fee for this. Certain books contain dozens of references while some articles have no references. You can't read every book and article in the limited time you have, but certain articles that directly relate to your research can be beneficial. As you delve deeper into your topic, you may need one of these other sources, and you'll be glad you took the time to set up the tree and copy some of the supporting articles. You can start reading your main article and doing your research without having the other articles in hand—just keep in mind that you might want to access the other articles at some point in the course of your research.

Beginning a Research Journal

The formal research paper will not be started until much reading and journal writing have taken place. The journal houses the "guts" of your research. It will be your guide to transforming your research into a research paper. The journal is a binder filled with lined paper, blank paper (for diagrams), and graph paper (for charts and graphs). A binder is recommended so you can easily remove pages, hand them in for checking, rearrange them, rewrite them, and so on. A binder can also house any other papers that can be three-hole-punched, such as computer printouts and copies of articles. For these reasons, a spiral or bound notebook is not practical for use as a research journal.

Perhaps you have kept journals for other courses; possibly even in other math courses. Often these journals are reflective—they record your reactions to and feelings about work that you've done. In such journals, being introspective may help you improve your skills because you are forced to analyze not only your mistakes but also the reasons you think you made them. This sensitivity can be tapped in future work. You'll remember how you felt about your performance because the act of writing it down clarified it.

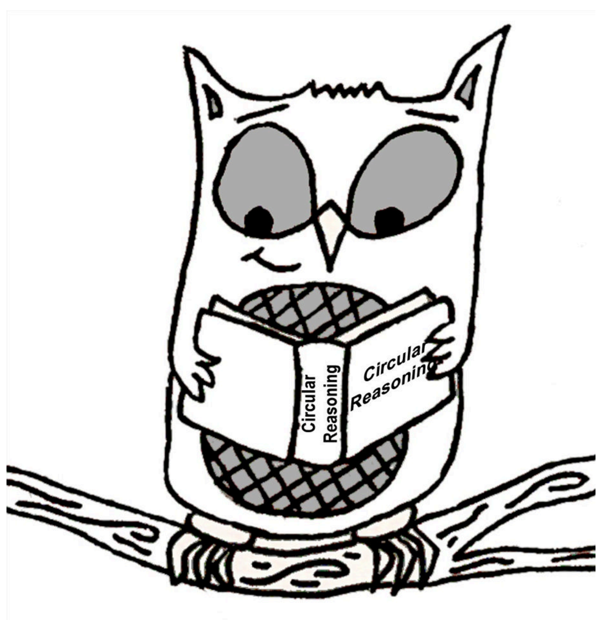
Your research journal, however, will not be a reflective piece. It will consist of the written work you create as you read your articles and take notes, test claims, make conjectures, try proofs, and so on. The journal must be organized. Pages of notes in the journal should be marked to correlate with the short notes on your annotation copy. Make sure you date the top of all your notes. Much of your journal writing will look like scratch notes—unpolished mathematical material you used to accentuate your readings. But be careful—you must be able to decipher these notes when you incorporate them into your final research paper. The meanings of short notes and abbreviations that you understood at the time your notes were written may be forgotten when you go through them months from now. Rewrite parts of your journal as you complete them so they are legible and thorough. Keep the notes in a logical order that follows the progression of your research. You can write up each section of your research as you finish it, using the writing tips in Chapter 5. Then it will be easier to merge them into your formal paper. A well-kept journal will translate efficiently into a research paper when the formal writing stage begins.

Apportioning the Readings

It is important to set up a research schedule and adhere to it. You should meet with an instructor or mentor periodically during the course of your research project. Perhaps you already meet with your instructor for extra help when you have questions about homework or class work. The extra help sessions for your paper will be called **consultations**. The project should proceed gradually. Meeting with your instructor provides time to ask questions, discuss the notes you take in your journal, and plan for the next reading. The consultations may take place as needed, but they should occur at least once every two weeks to keep the project progressing consistently. The amount of time spent at a consultation depends on the questions and material that need to be discussed. Before each consultation, read parts of your article and be prepared to discuss what you've read. The amount of time you have available, the natural breaks in the article's flow, and your instructor's advice should all be considered when deciding how much to read between consecutive consultations. A week's reading could consist of one page, one column, or one paragraph.

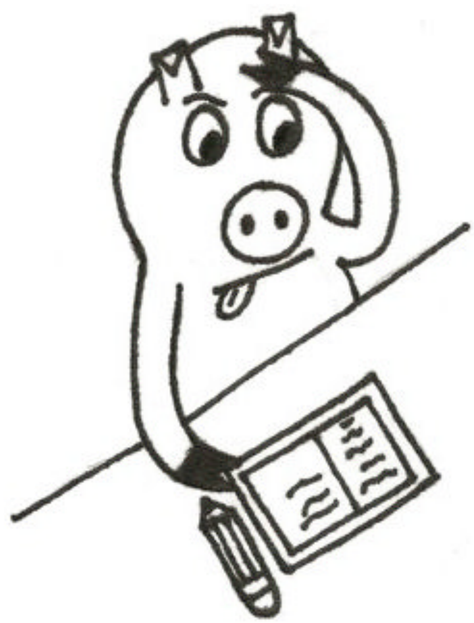
The work you prepare for a consultation is optimally done soon after the previous consultation. Why? The discussion you had with your instructor will be fresh in your mind, so it will take less time to do the work since you remember it well. It will also be better quality work. Once it is done, it won't be "hanging over your head" as part of a "to do" list. If you get stuck, you have ample time to get extra help or try and work it out yourself. If you get stuck the night before something is due, you have little recourse. Additionally, if you leave your work for the night before a consultation, and some unanticipated event arises, you have no time to get the work done. When you run into computer and printer problems the night before work is due, you also left yourself with no recourse. Being prepared early in your consultation cycle is a win-win situation!

Mastering a reading selection and extending it will take up much of your time, because these articles are written for math instructors. Part of your job will be processing the information and explaining it more comprehensively in your paper. Most of the articles will be three to six pages long. Theoretically, you could read the words on six pages in a few minutes. Before you begin your slow, careful, critical reading of an article, take time to skim it. Look at the section titles and the major theorems, diagrams, and tables. Read the conclusion. Make some mental notes about what looks familiar and what doesn't. After familiarizing yourself with the article, begin to read it more critically. Your mastery of the material will come as you read slowly, so don't proceed until you have mastered what you have already read. In some cases, you will need to read single words, sentences, and paragraphs several times before you understand them. Take time to think at the end of each paragraph or sentence. It may take you months just to get onto the second page of the article. Digest the information, and then think about possible questions, examples, conjectures, and additional material that relate to the reading passage.



Reading and Taking Notes

As you read, annotate your copy of the article by highlighting key wordsand phrases and by indicating where you have a question. You should testany claims that the article makes with examples of your own. Learn to read between the lines in mathematics. Because journal articles are written for people with deeper mathematical backgrounds than your own, steps are left out of proofs, and sometimes entire proofs are left out. Examples and diagrams are omitted and tables are condensed to save space in the article. Your finished paper will present the material in the article, along with original material, in a document that can be read and understood by your classmates with greater ease and in more depth than could the original article. Therefore, you have to supply the information that is implied "between the lines." Always read with paper, pencil, and highlighter in hand. You'll need to take notes that don't fit in the margins of your annotation copy. These notes will make your research journal grow quickly. Don't throw away notes—you might throw away something that could be of value later on. Don't forget to keep conjectures that you eventually prove to be incorrect.



We have spoken in general terms about some reading and note-taking skills. Now we take a look at a specific article, "On the Radii of Inscribed and Escribed Circles of Right Triangles," by David W. Hansen. This three-page article appeared in the September 1979 issue of *Mathematics Teacher*. We show the article here, complete with Sara's annotations.

ON THE RADII OF INSCRIBED AND ESCRIBED CIRCLES OF RIGHT TRIANGLES

Possible extension to equilateral? Isosceles right?

Mixing the Pythagorean theorem, the area of a triangle, and some first-year algebra yields some unexpected results.

By DAVID W. HANSEN
Monterey Peninsula College
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Also one circumscribed circle. Find its radius.

An *inscribed* circle of a triangle is a circle tangent to the three sides of the triangle with its center inside the triangle. An *escribed* circle of a triangle is a circle tangent to one side and to the extensions of the other two sides with its center outside the triangle. Every triangle thus has one inscribed and three escribed circles (see fig. 1). If we restrict our interest to right triangles only, an interesting relationship between the radii of the inscribed and escribed circles can be found.

Consider the radius of the inscribed circle of right triangle ABC as shown in figure 2. Let the lengths of the sides of the triangle be a, b, and c, with G the center of the inscribed circle, r its radius, and D, E, and F the points of tangency of the sides of the triangle with the inscribed circle G. Then, since a circle's radius is perpendicular to a tangent line at the point of tangency, the angles at D, E, and F are right angles, and using $m\triangle ABC$ to denote area, we have

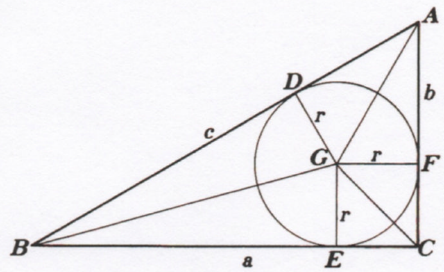

Fig. 2

"measure of the area"

$$m\triangle ABC = m\triangle BGC + m\triangle AGC + m\triangle AGB,$$

$$\left(\frac{1}{2}\right)ab = \left(\frac{1}{2}\right)ar + \left(\frac{1}{2}\right)br + \left(\frac{1}{2}\right)cr,$$

$$ab = r\,(a + b + c),$$ (Multiply by 2)

and

$$(1) \qquad r = \frac{ab}{a + b + c}.$$

Read to here first week.

Next, consider the radius of the escribed circle tangent to side a of right triangle ABC as shown in figure 3. Let G be the center of the escribed circle, r_a its radius, and D, E, and F the points of tangency of the extensions of b and c and side a with the escribed circle G. As before, the angles at the points of tangency are right angles, and

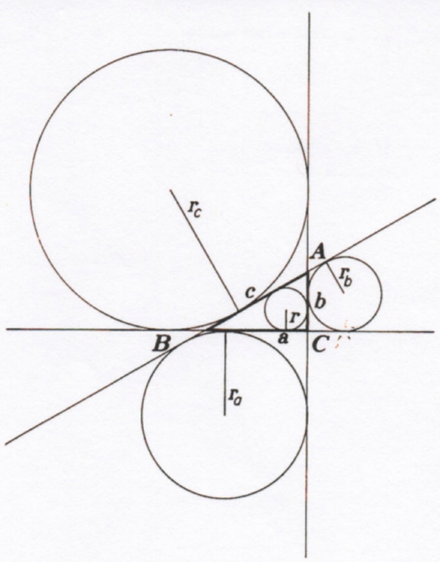

Fig. 1

[Show proof in full.] $\triangle ADG \cong \triangle AEG$, with $\triangle BFG \cong \triangle BEG$ by the hypotenuse-leg congruence theorem for right triangles. Thus, *[notation error]*

$$m\triangle ABC + m\triangle FCDG + m\triangle BEG + m\triangle BFG = m\triangle ADG + m\triangle AEG.$$

["Build" diagram in gradual stages.]

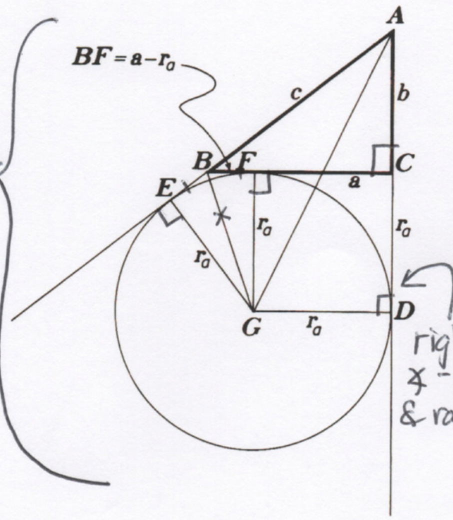

Fig. 3

[right ∡ – tangent & radius]

Because of the congruent triangles,

$$m\triangle ABC + m\triangle FCDG + 2(m\triangle BFG)$$

$$= 2(m\triangle ADG),$$

[FCDG is a quadrilateral, not a △.]

$$\left(\frac{1}{2}\right)ab + r_a^2 + 2\left(\frac{1}{2}\right)r_a(a - r_a)$$

$$= 2\left(\frac{1}{2}\right)r_a(b + r_a),$$

$$\left(\frac{1}{2}\right)ab + r_a^2 + r_a a - r_a^2 = r_a b + r_a^2.$$

[Read to here]

Now subtracting r_a^2 from both sides of the equation above and multiplying by two on both sides, we get

$$ab + 2r_a a - 2r_a^2 = 2r_a b.$$

Rearranging terms, we get

$$2r_a^2 + 2r_a b - 2r_a a - ab = 0.$$

Factoring r_a from the two middle terms of the equation above, we arrive at the quadratic equation (in r_a): *[coefficients for quadratic formula.]*

$$2r_a^2 + 2(b - a)r_a - ab = 0$$

Solving for r_a by using the quadratic formula, we get

$$r_a = \frac{-2(b - a) \pm \sqrt{4(b - a)^2 + 8ab}}{4}$$

$$= \frac{2(a - b) \pm \sqrt{4a^2 + 4b^2}}{4}$$

$$= \frac{2(a - b) \pm 2\sqrt{a^2 + b^2}}{4}.$$

[substitute]

Since $a^2 + b^2 = c^2$ by the Pythagorean theorem, we have

$$r_a = \frac{2(a - b) \pm 2\sqrt{c^2}}{4}$$

$$= \frac{2(a - b) \pm 2c}{4}$$

$$= \frac{a - b \pm c}{2}.$$

[Which sign? Why? Explain]

Since $c > a$ and $c > b$, we must have

(2) $$r_a = \frac{a - b + c}{2}.$$

(r_a would be negative if we used ^-c instead of ^+c.) *[Read to here]*

Consider now the radius r_b of the escribed circle tangent to side b of right triangle ABC in figure 1. By carrying out a procedure analogous to the one we have just done for determining r_a, we find that

(3) $$r_b = \frac{b - a + c}{2}.$$ *[(do entire derivation)] [Read to here]*

Finally, if we consider the radius r_c of the escribed circle tangent to side c of triangle ABC in figure 1, we find (using similar considerations to those used in determining r, r_a, and r_b) that

$$r_c = \frac{a + b \pm c}{2}.$$ *[(do entire derivation)] [read to here]*

Now, do we pick ^+c or ^-c? Figure 1 indicates that r_c is the largest of the four radii we are discussing. But, let's examine what happens if we pick ^-c in our formula for r_c. In that case, The difference of 2 squares, x^2-y^2, where $x=a+b$ and $y=c$.

Mult. by $\frac{a+b+c}{a+b+c}=1$

$$r_c = \frac{a+b-c}{2}$$

$$= \frac{(a+b-c)(a+b+c)}{2(a+b+c)}$$

Mult. numerator

$$= \frac{(a+b)^2 - c^2}{2(a+b+c)}$$

Square binomial

$$= \frac{a^2 + 2ab + b^2 - c^2}{2(a+b+c)}.$$

Then $a^2 + b^2 = c^2$ gives us

Pythagorean Thm.

$$r_c = \frac{2ab + c^2 - c^2}{2(a+b+c)}$$

$$= \frac{2ab}{2(a+b+c)}$$

or

$$r_c = \frac{ab}{a+b+c}.$$

But equation (1) tells us that the radius of the inscribed circle r equals

$$\frac{ab}{a+b+c}.$$

Now, this is impossible. Thus, ^+c *must* be selected in order to give the radius of the escribed rather than the inscribed circle. Thus, we have

Read ☆ to here

$$(4) \qquad r_c = \frac{a+b+c}{2}.$$

Summarizing our results, we have found that

$$r = \frac{ab}{a+b+c} = \frac{a+b-c}{2}$$

$$r_a = \frac{a-b+c}{2}$$

$$r_b = \frac{b-a+c}{2}$$

$$r_c = \frac{a+b+c}{2}.$$

Looking at the results above, we see a beautiful pattern of symmetry. Each radius equals one-half the sum or difference of the three sides of the triangle. Furthermore, r, r_a, and r_b each involve one subtraction, whereas r_c involves only addition, implying that r_c is larger than any of the other radii. But we can say more. By adding r, r_a, and r_b, we get

$$r + r_a + r_b = r_c.$$

Thus, we find that the sum of the radius of the inscribed circle and the radii of the two escribed circles tangent to the legs of a right triangle is equal to the radius of the escribed circle on the hypotenuse of the right triangle.

Show algebraic derivation

$$r_c = r + r_a + r_b.$$

But this is not all. If we multiply the smallest radius r by the largest radius r_c, we find that

$$rr_c = \frac{ab}{2},$$

Show alg. deriv.

which is the area of $\triangle ABC$. Also,

$$r_a r_b = \frac{ab}{2},$$

Show derivation

again the area of $\triangle ABC$.

Thus, we find that the area of a right triangle is equal to either (1) the product of its inscribed radius and the radius of the escribed circle on its hypotenuse or (2) the product of the two radii of the escribed circles on its two legs.

$$m\triangle ABC = rr_c = r_a r_b.$$

The calculations below illustrate our findings for a right triangle with sides 3, 4, and 5 (see fig. 1). Let $a = 4$, $b = 3$, and $c = 5$, then

$$r = \frac{4+3-5}{2} = 1$$

$$r_a = \frac{4-3+5}{2} = 3$$

$$r_b = \frac{3-4+5}{2} = 2$$

$$r_c = \frac{3+4+5}{2} = 6.$$

Make up other examples.

$$r_c = r + r_a + r_b = 1 + 3 + 2 = 6$$

$$rr_c = (1)(6) = (3)(2) = r_a r_b = 6$$

and the area of $\triangle ABC$ is $(1/2)(4)(3) = 6$.

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Long, Calvin T. *Elementary Introduction to Number Theory.* Boston: D. C. Heath & Company, 1965.

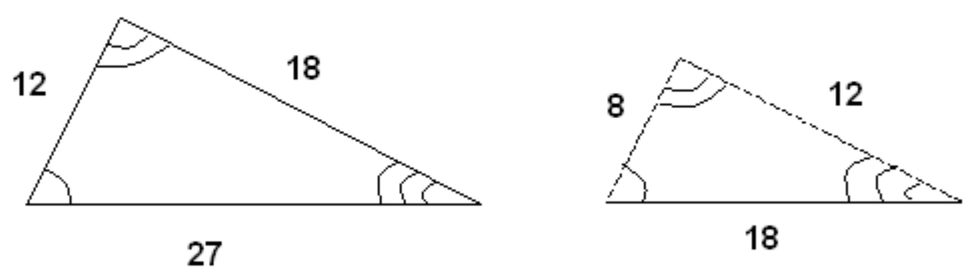
other possible sources

Let's examine some of Sara's annotations. Notice that the word *right* is circled in the title. The article is deliberately limited to claims about right triangles, and skimming over this single important word could mislead you. Throughout the article, notice the words and phrases Sara underlined. Can you tell why she underlined them? Keep in mind that even a single word or a short phrase can be significant. Sara wanted to call attention to words she considered important.

Before we continue examining Sara's annotations, let's look at how leaving out or adding a word can drastically change the meaning of a passage. How crucial can one word be? Read the following sentence and draw diagrams to test the claim. Determine whether the statement is true or false.

> If two triangles are similar and two unequal sides of the first triangle are congruent to two sides of the other triangle, the triangles are congruent.

The statement is false. The triangles are not necessarily congruent, because the pairs of congruent sides are not necessarily *corresponding* sides.

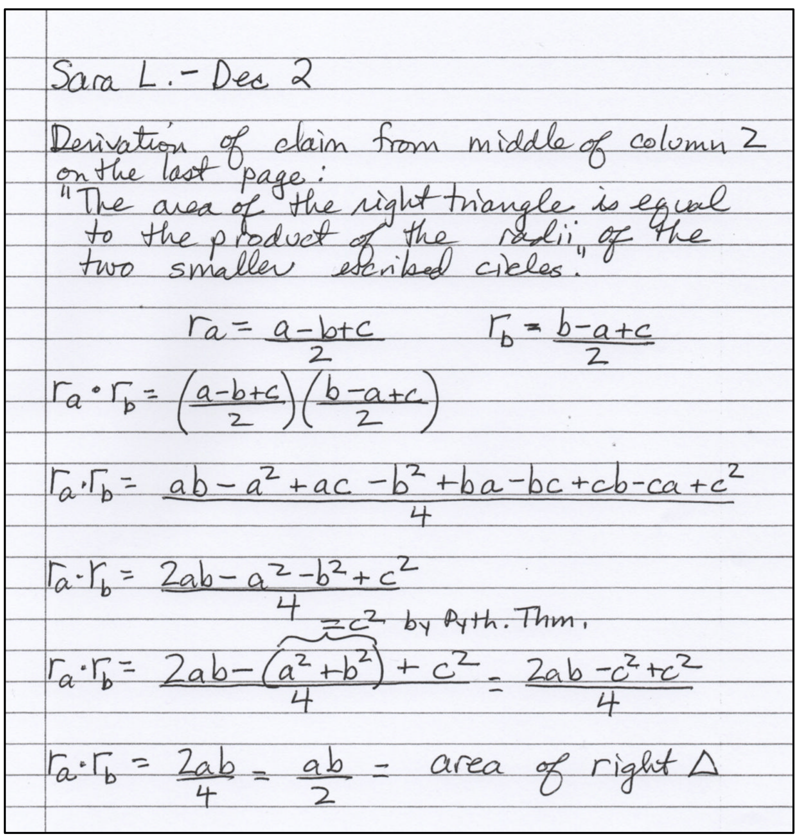


The triangles above are not congruent. If pairs of corresponding sides were equal, the triangles *would* be congruent. Readers who answer "true" usually inferred the word *corresponding*. The meaning of the statement is drastically altered by the inadvertent mental insertion of this word. As you can see, leaving out or adding one single word can lead to confusion. If Sara hadn't noted that the article dealt only with *right* triangles, she might have wasted a lot of time trying to apply the findings in the article to non-right triangles. Now you can see why Sara underlined so carefully.

Sara's knowledge of geometry led her to want to inquire about the circumscribed circle, as annotated in the first paragraph. In column 2 on the second page of the article, in the middle of the column, she notes an unfamiliar and potentially misleading notation for area. Notice that Sara's first reading assignment ended at the star in column 2. She will address some of the points she raised before she continues her reading.

On the second page of the article, Sara made some more notes. You can see that this page alone was divided into four different reading assignments. These were apportioned by Sara in a consultation with her instructor. Several claims are made that Sara must test and derive. At the top of column 1 is a theorem that Sara must state in full in her formal paper, so she will find the theorem and its proof before her next reading session. Notation for area in the middle of the column seems to label quadrilateral *FCDG* as a triangle. This never confused Sara, because she knew the intent. Sara plans to develop the diagram in column 1 in stages, using several diagrams with explanations in between each. She is also going to try to construct the radii of the circles using compass and straightedge. Toward the end of the page, claims are made that Sara must prove. Here she must read between the lines and supply original examples and derivations not given by the author.

The last page of the article shows more annotations that again require reading between the lines. All of the information Sara compiles becomes part of her research journal. As she accumulates information, she makes sure it is comprehensive and legible, so it can easily be incorporated into her formal paper at a later date. A sample page from Sara's journal is shown below. This page is her original derivation of a claim made in column 2 on the last page of the article. Take time to look at the sample page before continuing. Notice that this journal entry is clear and complete. Sara will have no trouble relearning this material when it is time for her to write this section of her paper.



Sara L. – Dec 2

Derivation of claim from middle of column 2 on the last page:

"The area of the right triangle is equal to the product of the radii of the two smaller escribed circles."

$$r_a = \frac{a-b+c}{2} \qquad r_b = \frac{b-a+c}{2}$$

$$r_a \cdot r_b = \left(\frac{a-b+c}{2}\right)\left(\frac{b-a+c}{2}\right)$$

$$r_a \cdot r_b = \frac{ab-a^2+ac-b^2+ba-bc+cb-ca+c^2}{4}$$

$$r_a \cdot r_b = \frac{2ab-a^2-b^2+c^2}{4}$$

$= c^2$ by Pyth. Thm.

$$r_a \cdot r_b = \frac{2ab-(a^2+b^2)+c^2}{4} = \frac{2ab-c^2+c^2}{4}$$

$$r_a \cdot r_b = \frac{2ab}{4} = \frac{ab}{2} = \text{area of right } \triangle$$

Sara could not find the books listed on her bibliography tree, so she looked for other articles. She found one on the Fibonacci sequence and the Pythagorean theorem. After looking at the article, she decided she could try to use some of the results to make new generalizations about special right triangles and escribed circles. The article showed how to generate Pythagorean triples using the Fibonacci sequence. Sara thought that if she examined right triangles with integer-length sides generated from the Fibonacci sequence, she might be able to express the radii of the circumscribed, inscribed, and escribed circles in terms of the original Fibonacci numbers used. Sara read and annotated this article and then attempted to combine the findings from both articles. Can you see how the original material Sara added to the first article and the combination of results with a second article made her paper a research paper, not just a math report? She began the formal write-up after two to four months of research. Her research continued during the writing process.

Conquering Difficult Concepts

Some parts of your articles will be more difficult than others to understand. If you don't agree with a finding, you might suspect that there is an error in the article. Notice that Sara's article used unconventional notation for area. Generally, journal articles are edited very carefully, and it is rare to find a mistake in an article. However, sharp students reading carefully and testing all claims have uncovered some errors.

Robin was reading an article on the Malfatti Problem in the May 1992 issue of *Mathematics and Informatics Quarterly*. An inequality expression erroneously had the inequality sign reversed. Robin e-mailed the editor (who was in Europe) and received a reply that, indeed, there was an error. Robin had spent hours trying to understand the incorrect inequality!

Megan was reading an article in the January 1986 issue of *Mathematics Teacher* on factoring quadratic equations with consecutive-integer coefficients. She found an error in one of the factorizations, traced the author to a college in Texas, and wrote to him. He replied, stating that, indeed, she was correct and that she was the first person to notice the error in the twelve years since the article had been published! Megan included these correspondences in the Appendix of her research paper.

Many times, readers who *think* they have found an error don't fully comprehend what they are reading. Upon rereading, trying examples, and so on, readers usually find that there are no errors. Occasionally, you may find an error, but don't assume that material you disagree with is in error. Read again carefully, and try some examples and diagrams to see if you can eliminate the problem. A cooperative effort might clear up the discrepancy. If another student is working from the same article, the two of you should get together and discuss the problem. If there are no other students working on your article, decide whether you can isolate the problem and explain its context to another student or to your instructor. Another person doesn't necessarily need to be familiar with the entire article to be able to help you out.

It is possible to move ahead with your reading even if you have questions about previous material. You might decide to accept the result you have questions about and read ahead, keeping in mind that the readings are based on the truth of the result you don't fully understand. Persistence, your problem-solving strategies, and your newfound reading and journal-keeping skills will go a long way in helping you decipher challenging material. Discussing the material at a consultation with your instructor will also help you master the tougher parts of your article.

Extensions Within and Extensions Beyond

Your paper will first lay the groundwork for your topic. This is where you do an in-depth study of the article you chose, adding examples and explanations along the way. Journal articles leave out many mathematical details and the authors expect that readers can "read in between the lines" mathematically. The target audience for these journal articles has an extensive mathematics background, most likely from their college educations. Some proofs and explanations are left out to save space, and others are left out because the readers do not need such explanations. In your paper, you can fill in all of these missing explanations. You can also test claims that are made in the article by making up numerical examples. You do not have any space limitations so you can give a comprehensive treatment of the entire article, including many more explanations than the original author did. If the author mentions the use of a theorem you never heard of, you might investigate the proof of that theorem. This will give you practice in reading and writing mathematics. All of these extensions fall under the umbrella of **extensions within**. Extensions within form a very suitable introduction into extending mathematics beyond what is printed on the page.

As you study your article, you may wonder how the topic can be altered, changed, extended, etc. The article's author may actually give challenges right in the article. The articles in your article's bibliography may have other related challenges and extensions. A Google search may also produce some ideas. These will comprise your **extensions beyond**. You will use your problem solving strategies to navigate through your extensions beyond. The extensions beyond phase is where you get to "play" with mathematics—try things, make conjectures, find counterexamples, create proofs, etc. This is not easy.

The polish of your finished journal articles and math texts do not show the errant trials and crumpled papers full of mistakes and dead ends. That is the reality of math research. Fermat, Euler and Gauss didn't wake up one day and sit down and decide to prove a new theorem. Most likely, their mathematical encounters revealed patterns and ideas to them that piqued their interest. They then, on their own time, decided to attempt and publish their results. Some of their findings took years to complete. Some simply went unfinished.

Everything you do mathematically as you study your paper should be included in the paper. Anything unfinished will end up in your final section, Recommendations for Further Research.

Frustration and dead ends come with the territory. It's all part of the journey, so embrace it. And remember, even finding a dead end gives useful information. You have warned future researchers not to "bother" to go there—you have already shown it "doesn't work."

So get it *all* down on paper!

The Journal Article Reading Assignment

Using Chapters 4-6, you completed a Math Author Project to gain experience in writing mathematics. This experience will help you take notes for your journal and write your formal research paper. Now that you have completed Chapter 8, you need experience in reading a mathematics article. Use the article you chose for your topic. Annotate the article, take notes, list questions, and so on as described in this chapter.

At this point, your research journal will consist of questions, examples, conjectures, theorems, patterns, derivations, proofs, and counterexamples. Over a period of months, you will add material to your research journal. All of this information will be compiled and presented in a logical fashion in a formal research paper that combines your research and your writing skills. Chapter 9 will help you assemble the components of your final paper.

Chapter 9

Components of Your Research Paper

When you write mathematics, your aim is to convey information as clearly and effectively as possible for a limited, specialized audience that we call the target audience. The target audience must have the requisite background to read the research. Precision, completeness, and clarity are paramount in mathematical writing—it can never be too clear. A finished mathematical piece will not read like a novel or a magazine article, and rightfully so. The purpose and the style are different. The poetic license you use as you create high-quality English essays is inappropriate in a math research paper. This is not to say that mathematical research papers cannot be enticing, riveting, or even suspenseful. You can arouse the curiosity of the reader by posing questions at the beginning of your paper that you will answer in your paper. The mission of the written paper is to transmit newly found results and to provoke the search for new knowledge. To the mathematician, the beauty of the findings is pure poetry. A research writer has succeeded if a member of the target audience becomes "hooked" by the intrigue of the opening questions and can then read, understand, and intelligently discuss the research. If your presentation is clear, future researchers can use it as a basis for further investigation in the field.

The writing and reading you have done all your life have contributed to your present writing ability. You will need to tap all the skills you have acquired as you put your paper together. In this chapter, we discuss the specific parts of your research paper. You should get a writing handbook from your school's library or English department. Use this handbook as a reference throughout the entire writing process. Keep in mind that writing follows formal sets of rules just as algebra does. Your research journal and your own memory will provide you with the mathematics material you need to build your paper.

The Structure of Your Research Paper

Before you write, you need a plan—you need to write an outline of your research paper. This outline will follow the order that your research takes, and will change as your research progresses. Before writing this outline, however, you should become familiar with the basic sections of your research paper, because your outline will be built around them. They are listed here in the order in which they will appear in your finished paper. You might not write the items in this order; we discuss why later in this chapter.

I. Cover Page
II. Abstract
III. Problem Statement
IV. Related and Original Research
V. Recommendations for Further Research
VI. References
VII. Appendix

We will discuss the function of each component of the paper in detail and examine a sample outline for a research paper. Keep in mind that you should develop drafts of all the components gradually, getting them corrected as you progress. When too much original writing is generated between checks by your instructor, it becomes more difficult for the instructor to read the paper and make the corrections. Additionally, you might be repeating errors that would have been found if you had had the benefit of an instructor's check. Hand in your drafts frequently, a few new pages at a time. You can include corrections you made on previously corrected material with your newly generated pages. If your instructor can edit the pages right in front of you, and discuss each comment with you, that would be an ideal method to strengthen your technical writing and your paper. The benefit is that you get an inside look at the editing process, and you get a feel for possible comments that will be made. You can then "anticipate" similar comments as you write, and this will improve your writing.

Cover Page

The cover page features the title of your paper. You might think of a title as you are doing your research, and you might revise this title several times as your work progresses. Why? The title should be descriptive; it may be longer than titles you've written for reports in other contexts. For example, Sara read the article featured in Chapter 7 on escribed circles to right triangles. She also read a second article that she notes in her annotations at the end of the first article. This second article describes how to use Fibonacci numbers to create Pythagorean triples. After completing the research of each article, Sara tried to combine results from the two articles and came up with a new, interesting result. She made this result the focus of her research, and it is reflected in her title. The title, as written in inverted pyramid form on her cover page, is shown here:

THE RELATIONSHIP BETWEEN THE SIDES OF A RIGHT TRIANGLE
GENERATED BY FIBONACCI NUMBERS AND THE
RADII OF ITS ESCRIBED CIRCLES

The following titles would not have been as informative, because they are not as descriptive:

RIGHT TRIANGLES AND THEIR CIRCLES

or

CIRCLES, RIGHT TRIANGLES AND FIBONACCI NUMBERS

Which title below do you think is more informative for the potential reader?

DERIVING THE EUCLIDEAN METHOD OF GENERATING
PRIMITIVE PYTHAGOREAN TRIPLES
or

THE PYTHAGOREAN THEOREM

The title appears in inverted pyramid form on the cover page. The cover page is also the place for your name, your affiliation, and the date of your paper. A sample cover page is shown as part of Michael's Math Author Project in Chapter 5 and as part of Dylan's paper in Chapter 11. The title itself should reflect the problem statement, because that is the major focus of the research. Your cover page will, in most cases, be written after your paper is completed. Keep in mind that its descriptive nature is linked to the problem statement. Recall from Tip 17 in Chapter 6 that you should avoid "cute" titles.

Problem Statement

The first section of a research paper is titled "Problem Statement." It includes an introduction that succinctly orients the reader to your topic. Make sure your introduction is appropriate for your target audience.

Your first paragraph deals with basic background material that the reader is already familiar with. This "invites" the reader in and familiarizes him or her with your paper's general area of mathematics. It doesn't intimidate the reader by immediately posing a challenge. Next, you begin to stretch the reader by "bridging" to the problem you are researching. You could hook the reader by teasing her or him with a few examples about your research problem. The reader might notice a pattern or formulate a conjecture based on your examples. If you inspire this, you have a good problem statement. Notice that all of this occurs before the problems are formally stated in a bulleted list.

Throughout the Problem Statement you might have to introduce new terms and define them. When the terms are first mentioned they should be in a bold font. Also, when defining new concepts, give some examples of the new idea, and some examples that do not fall under the new idea. For example, if you were defining 'prime number,' you would give examples of prime numbers and examples of numbers that are not prime numbers.

The reader needs some previous knowledge to follow your paper but should not become immersed in the complex specifics of your research until the stage has been set. Chapter 11 has samples of the Problem Statement section of a student's research paper. Note that the problem is stated early in the research paper. Although you must orient the reader, you should not give an excessive amount of background material before stating the problems. The questions posed in the Problem Statement section are the problems you will address in your paper. They could be problems posed in the article you read, original problems, or a combination of both. Look at Dylan's paper in Chapter 11 and determine how well you understand what the paper will be addressing. You can read other examples of problem statements in Chapter 11.

Related and Original Research

In Chapters 4, 5 and 6, you learned about the formal writing of mathematics. Tips, examples, and suggestions were given. Hopefully, you had a chance to practice some of these writing suggestions as part of a Math Author Project. (Your research journal entries may have been written more informally, since they were notes to be read by you only.) Use Chapters 4, 5 and 6 as references as you write the body of your paper, along with the suggestions given here.

Use headings to divide the paper into sections at logical junctures.

Headings were first discussed in Chapter 4. We review them here because they reflect parts of the research paper and are not as open-ended as the headings you created for the annotation project. Your outline can serve as a skeleton for constructing your paper. Devise a hierarchy for indicating new sections and divisions within sections. For example, you can use the techniques of justification, lower- and uppercase letters, and font to create a multilevel hierarchy of headings in your paper.

For example, let's look at one example of a possible hierarchy of divisions for your paper. The cues for each new section will come from the structure of your outline. Notice that the major section of the paper are centered and capitalized. The next level of division is left justified and underlined. Subsections of sections can be delineated by using a consistent hierarchy of section titles. These are often "tweaked" as you proofread. Here is one example.

PROBLEM STATEMENT

Triangles and Their Circles

RELATED AND ORIGINAL RESEARCH

The Radii of the Circumscribed and Inscribed Circles

The Radii of the Three Escribed Circles

The Escribed Circle to Side *a*

The Escribed Circle to Side *b*

The Escribed Circle to the Hypotenuse

Right Triangles and Fibonacci Numbers

Generating Sides from Consecutive Fibonacci Numbers

Formulas for Radii in Terms of Fibonacci Numbers

RECOMMENDATIONS FOR FURTHER RESEARCH

REFERENCES

APPENDIX

Changing fonts sizes and the use of bold print are other ways to vary the hierarchy. Just be deliberate in your choices and realize that the changes in title formats are not arbitrary.

Headings should never be placed at the bottom of a page; if your page break occurs right after a heading, move the heading to the top of the next page. If you have a consistent system for dividing up your paper, readers will clearly understand how the different sections and subsections of your research are related. There is a close relationship between the outline and the section divisions of your paper.

Divide each section pragmatically into paragraphs with respect to mathematical content development.

Remember to center and highlight mathematical expressions that need to be "digested." Recall from Chapter 3 that the page layout of your paper will be visually different from that of a novel. Present your material with logical breaks indicated by new paragraphs. Make sure your paragraphs and sections are logically connected to those that follow. Use sentences (segues) at the end and beginning of sections that explain why headings occur at their specific juncture in the paper. Whenever possible, end sections, subsections, and paragraphs with a motivating idea that is addressed by the next section or paragraph.

For example, suppose you were writing a paper on the quadratic formula. You would discuss roots, factors, trinomials, and binomials and possibly even show a geometric interpretation of some of your major points. You would start with trinomials that were factorable. If section three of your paper develops a derivation of the quadratic formula, you can end section two with your paper's first mention of a quadratic equation with a trinomial that can't be factored, such as

$$x^2 - 8x - 5 = 0.$$

After having "indulged" the reader in solving quadratic equations by factoring, you have now motivated the need for another method, *because factoring doesn't work!* After reading and understanding your research on the factorable equations, the reader *wants* to know how to solve the problem you've posed. Now your treatment of the quadratic formula is not arbitrary but driven by need, creating a solid flow between these two sections of your paper. When you proofread, determine whether your paragraphs and sections flow sensibly. You can add transition sentences (segues) as you proofread to smooth out the flow of the paper.

Credit authors if you use their findings, even if you use no direct quotes.

These credits are called **citations.** Include the author's name and year of publication in any citation. If interested, your reader can find complete information about a source in the References section of your paper. You should look up how to note citations in your writing handbook. Here are some examples:

Trobiano (2009) found that any whole number can be expressed as . . .

or

In 2009, Trobiano found that any whole number can be expressed as . . .

or

Any whole number can be expressed as . . . (Trobiano, 2009).

The full name and the source can be found in the References section if the reader wants more information on the original source. Consult your English teacher and/or Internet sources for help in formatting citations.

Follow the rules you learned in an English or composition course for direct quotations, or use a writing handbook.

Make sure you include at least the author's last name and the year the material was published. Short quotes usually require quotation marks; longer quotes (forty or more words) are sometimes indented in their own block of text, without quotation marks. Consult your English teacher and/or Internet sources for help in formatting direct quotations.

Recommendations for Further Research

Recall from Chapter 1, or from your own research experience, that each new venture in mathematics raises many questions. In your research, you found some answers, and you may have wondered about the answers to questions that arose in relation to your research. These questions might fall outside the narrow focus of your research, but they can be posed for other researchers to pursue. They are logical next steps based on the conclusions stated in your paper and could actually become the problem statements for another research paper. Chapter 11 has examples of Recommendations for Further Research. This concluding component of your paper is *not* a summary of your findings, but the introduction at the beginning of this section can, in 1-3 sentences, concisely recap what you've done. Notice that this is a short section.

Abstract

When your paper and cover page are completed, you can write an **abstract.** An abstract is a short summary of what the reader can expect to find in your paper. It appears right after the cover page. An abstract is not an introduction—it is a summary. It should be approximately 250 words or less, on a separate page headed "ABSTRACT." The abstract should contain one or two paragraphs and be single-spaced, centered between the top and bottom margins. It shouldn't go into great detail but should convey the essence of your research. An abstract allows future researchers to quickly determine whether your research might be of interest to them. People read abstracts to determine if they want to purchase a book or download a paper.

Below are two examples of students' abstracts. The first is Tim's abstract from his paper on Bobillier's theorem.

> Bobillier's theorem states that "if a triangle of fixed size moves in the plane in such a way that two of its sides are tangent to two fixed circles, then the third side will be tangent to another fixed circle." This research develops a proof of the theorem. Relationships between circles and tangents, and circles and triangles, are explored to give an overview of the concepts necessary to understand the proof of the theorem. A lemma (a theorem whose primary function is in the proof of another theorem) for Bobillier's theorem is proved, making use of these concepts, and the theorem itself is

> proved. Suggestions for further research concerning the theorem are included at the end. These include possibly using other polygons and non-convex polygons in place of a triangle.

Daniele's research on probability and the quadratic formula is summarized in her abstract.

> This research explores a purely mathematical question and analyzes its solution. This research investigates the probability that a quadratic equation $y = ax^2 \pm bx \pm c = 0$, where b and c are randomly selected real numbers and $a = 1$, has real roots. The problem is approached through a number of different methods and techniques, some more accurate and efficient than others. One method is the Monte Carlo method, which uses area relationships to find probabilities. The probability of the roots being real is verified using integral calculus. In finding the solution, the research relies on conic sections, calculus, functions, geometry, and probability.

Other sample abstracts can be found as part of Michael's Math Author Project in Chapter 4 or in Dylan's paper in Chapter 11.. Because the abstract is a summary, writing it is the last step in writing the research paper. So, the cover page and the abstract, the first pages of the final document, are actually written last! A well written cover page and abstract establish that the research to follow is serious, high-quality material.

References

The end of your paper will need a reference list, or bibliography. In many cases, this will include only the articles you read. You must include any article that you cite in your paper. There are several different styles that your references can follow. Use the style you were taught in an English or composition course, or adopt another form from a writing handbook or Internet source. Your librarian may also be able to help you. You can also follow the form used in the reference sections of the books and articles you've read. Some of your sources might be Internet Web sites. Be sure to note them as well, in correct form. Start by taking a look at the References section of this book.

Appendix

Under certain circumstances, your paper may need an Appendix. The Appendix appears at the end of the paper and houses materials that should not appear in the body of the paper, mainly due to their length. What information might comprise your Appendix? You may have used large sets of raw data or lengthy lists in your paper. If you executed a computer program, you might want to include a copy of the program and its output. You can explain how the program works line by line. If the program is not original, be sure to give credit to the programmer. If your research involved the use of forms, questionnaires, or surveys, include these in your Appendix. If you corresponded with the author of an article, include all letters. If you interviewed someone for your research, you can include a transcript of the interview. Pertinent newspaper and magazine clippings can also appear in the Appendix. The Appendix is the correct place for items that are too cumbersome for the body of the paper. You can refer to the Appendix in the body of your paper. Not all papers will have an Appendix.

Creating an Outline

The outline of your research paper should present the material in a logical order. Often the order is based on the order of presentation in your readings. Your readings may include claims and proofs that skip steps or combine steps, because the intended audience of the article is someone with a mathematics background. You can add some original statements, reasons, and examples to improve the article's "abridged version." (We previously called this "reading between the lines.") You will have to decide where your extensions and "between the lines" original material fit. Be sure to include all of this material in your outline. You will need to revise your outline if, during the course of your work, you decide to change the material you present or the order of its presentation. The outline is for *you*; its function is to be a helpful guide. It is not part of the paper, so feel free to adapt it to suit your needs. It will grow and change as the research progresses, so it is not actually "complete" until the paper is finished.

Putting the Parts Together

You will be "playing" with mathematics for weeks before you actually start writing your paper. You will be reading, testing claim in your readings, trying new things, giving examples and counterexamples, and trying to generate and understand written proofs. You need to see the "big picture" as to where your research is generally headed for you to introduce it effectively. So the initial stages are handwritten pages in your research journal based on your readings. If you take good notes, you will be able to turn them into a cohesive paper more easily. You may actually start formally writing up parts of your Original and Related Research section before completing your Problem Statement. Word processing has made editing and proofreading much easier. Can you imagine what it was like to put together a research paper using a typewriter?!

As you complete pages of your writing, you will read them over and review them. You can avoid submitting drafts with many errors by using the suggestions in this chapter and Chapter 3 as a checklist when you proofread and edit your work. Chapter 11 has additional student sample pages you can use for guidance, and an abridged student paper as well.

Proofread and edit every draft before submission. Address all of your instructor's comments from previous drafts. Your instructor may have some examples of former or in-progress student papers you can look over.

As your work progresses throughout the term, you will develop valuable research skills, including reading, writing, conjecturing, problem solving, and proving theorems. If, during the course of your education, you write more math research papers, you will grow in your experience of mathematics as an inductive, discovery-driven science. You might decide to enter a mathematics contest. If you are interested in entering your research paper in a contest, read the next section and discuss your options with your instructor, the math department chairperson, or an administrator. You may even decide to start your own Math Research Symposium in you own school!

Entering Math Contests

Your experience with reading, writing, and research has made you a better researcher. These skills will also help you become a better problem solver. You are now better equipped to read problems, extend problems, and write well- explained solutions to problems. If you enjoy problem solving and/or math research, there are several contests opento students just like you!

Contests can be school-wide, countywide, statewide, national, or even international. They may involve problem solving or writing a research paper. Some require teamwork, and others require individual entries. Some require an extensive written solution to a problem, which is sometimes open-ended. The solution may even require that you build something. Some contests are in the form of scholarship exams. Contests will tap every facet of your problem-solving and math research ability. They provide a unique forum for you to blend communication, reasoning, connections, problem solving, and technology with mathematics. Prizes for contest winners range from plaques to trophies, trips, books, cash, and even college scholarships. Not everybody wins a contest but in many ways all entrants are winners. When you complete a project that really tested your mettle, made you work to your potential, and instilled knowledge in you, you truly are a winner!

Each year there are many contests, some new and some perennial favorites. At certain times, some contests may be discontinued. To keep abreast of the currently available contests, you can contact the organizations listed in Chapter 13 by doing an Internet search for their most up-to-date contact information. Contact your local and state mathematics teachers associations for a list of contests in your area. Many contests that, by name, appear to be *science* contests include mathematics as a science. A list of some major contests appears in Chapter 13. In some cases, you can apply for information directly. In other cases, your instructor should send for the information. Talk to your instructor about securing information about each contest; then you can decide which are for you.

Read the rules carefully for any contest you plan to enter. Contests involving research papers may have spacing requirements and a page minimum and/or maximum. There could be other regulations that require you to go back to the computer and adjust your paper to conform to the required form. Don’t disqualify a fine effort because you didn’t follow directions. Read and scrutinize!

If the prospect of a contest deadline seems a bit overwhelming now, prepare for the future. Send for the application packets and become familiar with the expectations of each contest. You may decide to enter at another time. You might work on your research paper this term and decide to continue it next term. The summer is a great time to make headway on research papers. Many universities offer summer programs for high school students, and many of these programs provide support services for students involved in research. Contact your local colleges to find out if they offer such a program. For information on national and international programs, look in the journals and newsletters of mathematics and education organizations or do an Internet search to request information on summer programs.

Some contests require the entrant to make an oral presentation in addition to submitting written work. Chapter 10 will help you put together an oral report that reflects the high quality of a well-done research paper.

Chapter 10

Oral Presentations

At certain times in your life, you will be judged not solely on your credentials or on your written work but on your performance at a "live audition." Certainly, a job interview is a classic case of this. You send in a resume, academic transcripts, and references for a job, all in the hope of gaining an interview. If you are granted an interview, then you are qualified, on paper, for the job. The interview allows a potential employer to see how you carry yourself, that is, how you answer questions and handle discussions. A similar scenario may take place in college interviews. Many people are nervous in an interview situation.

Public speaking also makes some people nervous. Have you ever given a speech to a large group? Most people don't do this often and therefore don't get much practice at it. Presenting scholarly material requires knowledge as well as an effective spoken delivery. You will have to answer questions. The need to think on your feet, without a prepared script, makes giving a good live academic presentation both a challenge and a very rewarding experience. It is difficult to simulate the exact conditions of such a presentation, so most practice sessions are done without an audience. If you plan to make an oral presentation on your math research, what steps can you take to make sure you look professional, even if you've had little practice speaking in front of a group? This chapter will help you methodically prepare for your oral presentation so that you can deliver it with confidence.

P^4: Planning, Preparing, and Practicing the Presentation

Your math research paper represents months of work. You may already know much of it off the top of your head because you have internalized it by reading carefully, discussing the material at consultations, working through original material, and frequently revising your written presentation. A command of the material in your paper is the first step in building your oral presentation. With this as a foundation, your next goal is getting your point across to your audience. Start by planning your oral presentation.

Planning Your Oral Presentation

Two key factors will affect how your research is presented. First, you must know your audience. The presentation must be adjusted to appropriately address the backgrounds and mathematical capabilities of the people you are speaking to. Imagine how the presentations on the same math research paper might differ for each of the following audiences:

- your math instructor
- your English instructor
- the parent–teacher association
- a contest judge who is a college math professor
- the students in your math class

The second key factor that affects your presentation is the amount of time you are allotted. The difference between what you can cover in fifteen minutes and what you can cover in thirty or forty minutes is astounding. You may have difficulty cutting your presentation to the allotted time. At other times, two minutes can seem like an eternity. Compressing your work into a given time slot requires that you edit the paper and look for material that can be cut or condensed. The material you can leave out and the depth of the material you present must be considered carefully. If you are entering a contest, be sure to note all of the regulations and requirements regarding the oral presentation. If there a specific rubric that the judges use, see if a copy is available so you will know what the judges are looking for. Make sure you are aware of what percent of your evaluation is based on the paper and what percent is based on the oral presentation.

In the planning stage, you must focus on the content of your presentation. Your audience will benefit most if you customize your presentation for them. Generally speaking, the material will be presented in the order in which it appears in your paper. The mathematical content of your presentation can be divided into three parts:

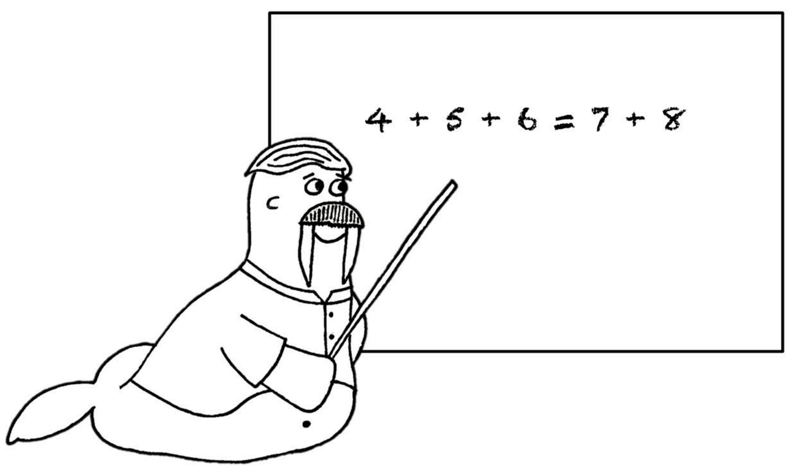
- introduction
- body of presentation
- conclusion

Let's discuss each of these parts in detail.

Introduction. When you plan your talk, remember that you should first give your name and the math course you are currently enrolled in. This will orient your audience. State the topic of your paper and discuss how you found your research topic. Do not memorize and then recite the name of your paper, since the titles are usually long—just informally give the topic you will be presenting.

You should prepare a one-page handout for the audience to engage them. We'll call this handout a "teaser" since it is designed to tantalize, engage, and orient the audience. The teaser should not be summary information about your paper—it should be a short question or challenge related to your research that will hook the audience. It should be able to be done by a member of your target audience in under two minutes. It may a variation of your problem statement. It could be from any another part of your paper. It could involve the audience in finding a pattern, using a manipulative, or solving a simpler problem related to your research. The idea is to orient all members of your audience. Make sure the title of your paper and your name, school and grade level are centered on the top. Unlike the research paper itself, you can have a little fun with the teaser by adding some cute artwork, livening up the fonts, and so on. At the bottom of the teaser, after the orientation exercise you present, should be your abstract. You might distribute the teaser while you are setting up the visual aids for your presentation, before you actually start your talk.

A sample handout from a paper on consecutive integer sums follows. Notice that you don't need to know about consecutive integer sums to understand the handout. In this case, the handout acts as a bridge from simple addition, with which most people are familiar, to the paper's topic. Try the handout yourself. With a little trial and error, you will get the answers to the first five examples given.



There is no consecutive- integer sum for 8, so 8 was deliberately placed last to "playfully frustrate" the audience and hook them for the presentation. The reason 8 does not have a consecutive-integer sum is addressed toward the end of the presentation. The answer to your handout should be addressed at an appropriate juncture in the oral presentation; not necessarily at the beginning.

AN EXPLORATION OF RELATIONSHIPS BETWEEN
CONSECUTIVE INTEGER SUMS
AND THEIR ODD FACTORS

Sheldon Wallowitz
South Falls High School
Grade 9

The numbers that are added in any addition problem are called the **addends**. A **consecutive integer sum** for a given positive integer *n* is a set of consecutive positive integer addends that add to *n*. For example, a consecutive integer sum for 10 is 1 + 2 + 3 + 4 since

$$1 + 2 + 3 + 4 = 10.$$

Find a consecutive integer sum for each of the following integers:

17 = ______________________________

20 = ______________________________

6 = ______________________________

8 = ______________________________

ABSTRACT

A **consecutive integer sum** for a given positive integer *n* is a set of consecutive positive integer addends that add to *n*. This paper investigates relationships between consecutive integers sums of an integer and the odd factors of that integer. First, consecutive integers sums are found using trial and error. Then, a method for finding consecutive integer sums in introduced. If a number has a consecutive integer sum, then it has an odd factor other than 1. The converse of this theorem is also proven. Recommendations for further research include relating the number of consecutive integer sums to the number of odd factors.

Body of Presentation. After the introduction, you present the major points of the article or articles you read, and your trials, findings, conjectures, etc., in a natural order that is usually the order the concepts were presented in, in your paper. You need to cover essential background material. Lead the audience through the material's major points as if you are discovering the mathematics with them as you proceed. Make conjectures as you proceed based on the information you have presented. Items developed in your paper through patterns and examples that led to conjectures and proofs should be presented that way. By not merely stating theorems, you will rouse the mathematical curiosity of your audience. Don't just recite facts—ask questions and let the audience see the material that led to the conjectures and claims you eventually made. In other parts of your talk, answer the questions you have posed.

The key here is *empathy*—you have been working on your paper for months and many facets of it are second nature to you. You need to simulate, to yourself, the background of your audience, which was the background you had before you started your research. Being able to explain something you are knowledgeable about to someone who is a novice is a real art, and takes careful thought and practice. Be sure to define new terms and give examples of them when they are introduced.

You don't want to "spill the beans" too early—you want the audience mentally at the edge of their seats waiting for your explanations of the questions you pose during the body of your talk. Yes, a math presentation can have mystery and intrigue, just like a good movie!

If there are several similar proofs in your article, don't go through all of them in detail in your oral presentation. Pick only one to explain. For example, Sara gave a presentation of her paper, which was based on the article on escribed circles in Chapter 8. The article featured the derivation of the length of the radius for one circle. Lengths for the radii of the other two circles were given but not derived. Sara derived them in her paper. Since the derivations were similar, she presented only one proof in her oral presentation.

Much of the original work in your paper may be interspersed with your survey of the article or articles, since you added extensions within and extensions beyond. The original work might not be easily separated from the article's findings, and that's fine. However, often your extensions beyond can clearly be delineated in your presentation as original findings. For example, Jocelyn's spreadsheet in Chapter 6 represents an extension of her article. Her article focused on area, and she investigated perimeter after being inspired by the article's findings. Your original investigation can include conjectures as well as original proofs. If you read several articles and tied the results together, include this combination of results as an original finding. Your oral presentation should stress your original conjectures and discoveries as much as possible, since that is the essence of research.

Conclusion. The conclusion of your presentation is not a summary of your paper. You can briefly summarize the major points of your paper as an introduction to your recommendations for further research. Your presentation concludes as your paper ended, with these recommendations. Present the recommendations from your paper and explain why they are a logical next step based on the research you have completed.

The amount of time you devote to each section depends on your particular research. As the expert on your topic, you need to make decisions on what to include. If most of your research focused on an article, then your section on original findings might be very short. If you made elaborate conjectures after reading an article, a majority of your presentation may deal with the original findings. The presentation is heavily dependent the time allotment. Revise the presentation as necessary during the practice stage of your oral presentation development.

Keep in mind that "less is more." Rushing and talking quickly to get more material into your presentation indicates poor presentation skills. Cover less, at a good pace, and cover it expertly. Empathize. Pause frequently so the listener can digest the information you just presented.

Visual aids and ancillary materials can be used to make significant time adjustments, because pictures can covey information so concisely and eloquently. The next section will discuss different visual aids you can incorporate. Later in this chapter, you will read about coordinating your presentation with your time allotment and the visual aids you selected.

Preparing Materials for Your Oral Presentation

You will, of course, need a copy of your paper for your use during your presentation. Judges, other students, or your instructor may ask you questions about items in your paper, and you will need to refer to it. You should try to anticipate questions and prepare the answers. You can tab this copy of your paper with sticky notes and use the tabs to quickly find crucial parts of your paper. You can also annotate the paper with notes that may help you wherever necessary.

Before you create any ancillary materials for your presentation, be sure each one has a specific purpose. Employ art only to enhance mathematical understanding. The artistic quality of any visual aid you produce should be superseded by its effectiveness in transmitting information crucial to your report. It is hard to imagine a mathematics presentation that does not use visual aids. If a picture is worth 1000 words, then visual aids can transmit information in an efficient, time-saving manner.

There are several different visual aids you can tap as needed:

- PowerPoint/Keynote/Prezi and other presentation software packages
- chalkboards, white boards, or interactive computer-driven whiteboards

- posters
- tri-fold displays
- overhead transparencies
- manipulatives
- models
- computers
- graphing calculator emulator software
- video
- felt boards

Careful use of these visual aids will help transcend your presentation form "tell" to "sell." What does it mean to "sell" a presentation? It means you are academically and emotionally engaging your audience and convincing them that you are invested in the topic and that you enjoyed studying it. *Telling* involves words—*selling* involves passion.

Choose your visual aids with discretion. Consider supplying your audience with some of these visual aids, or photos of them in use, in a handout so they can follow your presentation. You might even make manipulatives and models available to your audience. We will discuss each of the visual aids listed above in detail.

PowerPoint and Keynote Haiku Deck, Prezi, Google Drive Presentation. If you have access to a laptop and a projector, you can prepare your entire oral presentation using presentation software. PowerPoint, Keynote, Prezi and others allow you to design a custom video presentation. You can build graduated diagrams one stage at a time, and you can unfurl lines of algebra line by line, as they are explained. The possibilities are endless. Creating the slides is fairly simple, once you get the knack of it. Talk to your math instructor or a computer instructor to find out more about using presentation software. Check out what type of presentation software your computer has. Make sure you have equipment suitable for your audience and room size. Bring a pointer so you can show specific parts of the slides without having your body block the view. Presentation software usually has its own built in electronic pointer.

Remember that clarity is your mission—do not get involved with making something glitzy for no reason. It is tempting to play with backgrounds and fancy transitions. Keep in mind this is not a "memories" video for a Sweet Sixteen party! Be creative, but be sensible. Keep the background plain white, keep the font simple and academic, and use color with purpose. Each slide should have a title. If a topic runs into several slides, the title can be the same, with the word 'continued' added to it.

You can build intricate geometric diagrams by animating in one line, shape, or angle at a time while you are explain the changes. You can add lines to a list one at a time to show a pattern developing. You can transform algebraic equations one step at a time by highlighting terms and having a subsequent line show the new form of the equation. Your imagination will be a tremendous help!

Use divider slides to separate your presentation into sections. Make a bullet list of all the sections on one slide. As you transition from section to section, bring back that same slide, with one alteration. Highlight, in a different color, the section you are transitioning to. So, if your presentation has five sections, there will be five divider slides. In each slide a different section will be highlighted. This gives the audience a global perspective of your presentation.

Print your slides as handouts, 4-6 per page, and use these print outs to do additional editing and/or annotations for the presentation. Put them in a binder with your paper, to bring to your presentation. When presenting, do not read your slides. This turns the technology into an electronic cue card. It is uninteresting for the audience, and raises questions about your mastery of the material. You may decide to put running times in the margins of the slides, that the audience does not see, to make sure you are timing your presentation well. You can even pull up a stopwatch on your computer to help you keep good timing. Some software has a 'presentation mode' that allows you to see the next slide in the presentation. This allows you to formulate better transitions as you present.

The key is to practice the presentation often, in front of an audience whenever you can. This could be parents, friends, teachers, etc. Practice and revise! Presentation software often keeps track of the time spent creating the presentation. Don't be surprised if you cumulatively spent dozens of hours creating a 15-minute presentation!

Chalkboards or marker boards. You'd be surprised how a little planning can make a simple presentation on the board look professional. Instructors are used to writing on boards; as a student, you probably are not. You need to make a board map—a schematic of everything you will actually write and where you will write it. Avoid the tendency to write arbitrarily in any available space, write content out of order, erase material prematurely, and write too small or too large. Students who make board maps will impress the audience with their obvious planning and polish. Start writing in the upper left corner. Move down and when you are near the bottom move across to the right and start at the top again. Colored markers and colored chalk are great when you need to highlight something. You can even project a PowerPoint image on a board and then write directly on the projected image to make some mathematical point clearer to your audience.

"Live" board work that comes "right from your head" is a very good way to show your interest and command of the material. It makes you look authentically involved—it helps *sell* the presentation to the audience.

Posters. You will need posters for material that needs to stay in view, because as certain slides change, you might still need some information, for example, formulas or a geometric diagram, in full view during the progression of several slides.

Posters can be used if the distance between the presenter and the last row is less than 30 feet. A poster could not be used in a room 100 feet long. Letters on posters designed to be viewed at 30 feet must be over 1 inch high. Before you put any work into your posters, view some sample font sizes from the distance your audience will view them.

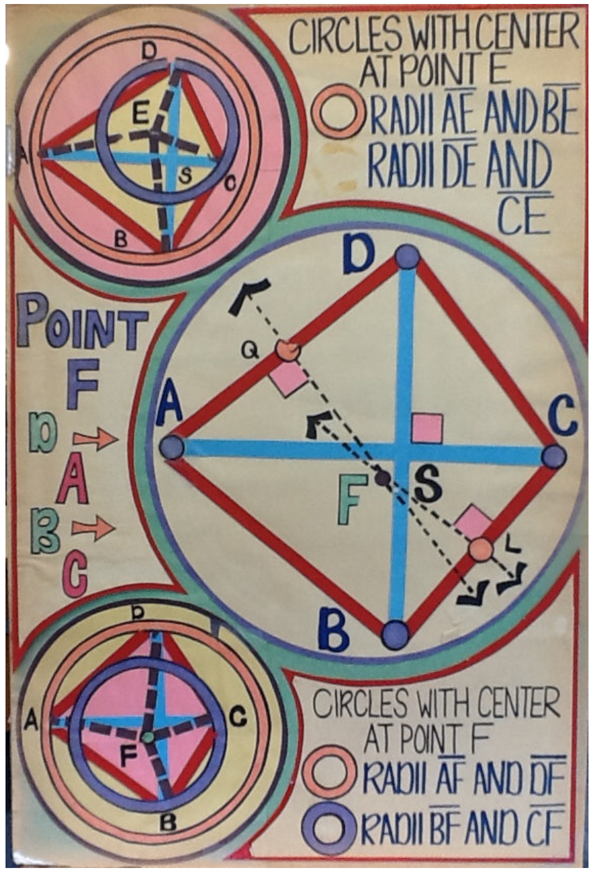
How do you decide what material to put on a poster? Posters are for material that must stay in view—that the audience may need to mull over while you are addressing them, and possibly even after you address them and move on to other slides.

As you read through your presentation outline and your research paper, decide what portions lend themselves to a poster treatment. Sketch your posters on paper first. Proofread your posters as you create them. When you prepare your posters, make sure they can be easily seen by everybody in the audience. If the size of the audience necessitates a large room, posters may be of little value to people in the back rows. This is usually not a problem with math contests, however. In most cases, your presentation will be in a room suitable for the use of posters.

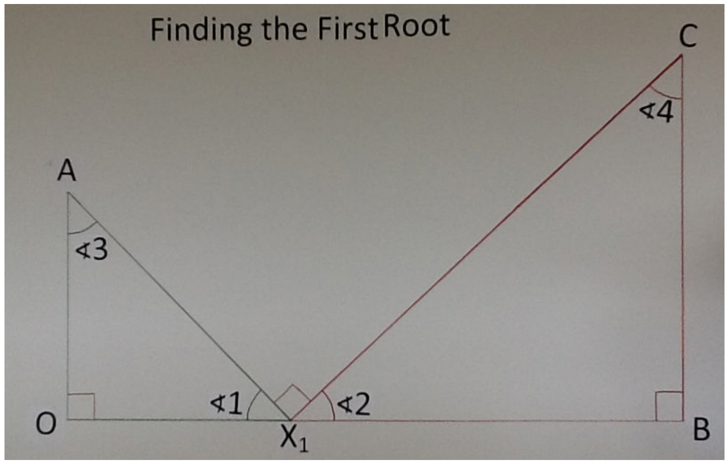
The computer and the photocopier work well together to provide professional results. You can set up your text on the computer—including diagrams, tables, and text taken straight from your paper—enlarge them on a copy machine, and paste them up on your poster. For some applications, characters generated by a large font size on the computer may be sufficient, and you may not need to enlarge the printout. Choose a suitable font—no script, and nothing fancy. In general, use color only to help illustrate a point; do not use color for color's sake. Many of your posters may be black and white. You can bring files on a flash drive to office supply stores that have printers that can print on poster-sized paper.

If you are artistic, and have neat handwriting, you can create most of your posters by hand. This is actually more time consuming than pasting up words generated on your computer. Remember, as in your paper, clarity is paramount. Keep your posters as uncluttered as possible.

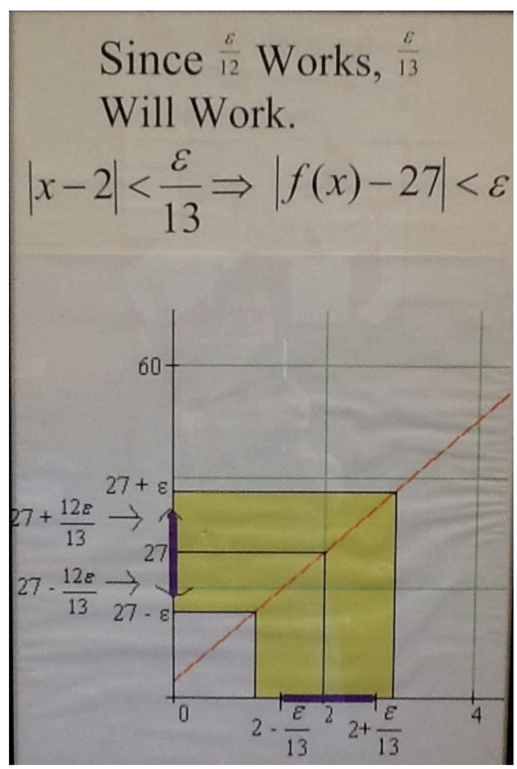
The poster shown below is a marvelous work of art that took a great deal of time, talent, and effort. However, the intricacy of the art and color made it more difficult to digest the mathematical points the presenter was trying to make. So it took more time and effort to generate a less effective product. If you print on white paper, and use a white poster board, when you glue your words to the board, they will almost look as if they are printed right on the board. Office supply stores will also print any file you bring in on paper as wide as three feet!



Posters should be done on foam-core (a foam-filled paper "sandwich"), which is available at art supply and stationery stores. Oaktag and mat board are not as good but may be used. Oaktag and mat board do not stay flat, and this creates problems at presentations. Foam-core is perfect when you have a ledge, such as a blackboard ledge, to stand your posters on, rather than securing them with tape, because then you can use the reverse side as well during your presentation. If you use both sides, make sure you know what is on the reverse side of each poster. You can put a little pencil mark in the corner to remind you. If two posters need to be displayed at the same time, make sure you do not put these on the front and back of the same board. Foam-core comes in white and colors. You can draw on them, paste things on them, and so on. Foam core posters fit neatly on a blackboard ledge. If there is no ledge, you can apply a small piece of double-stick foam tape, or even a magnet with adhesive on one side, to the back of each poster. Your posters will now stick to the board but will be easy to remove when necessary. Take a look at the simple, clean lines in Ayanna's poster.

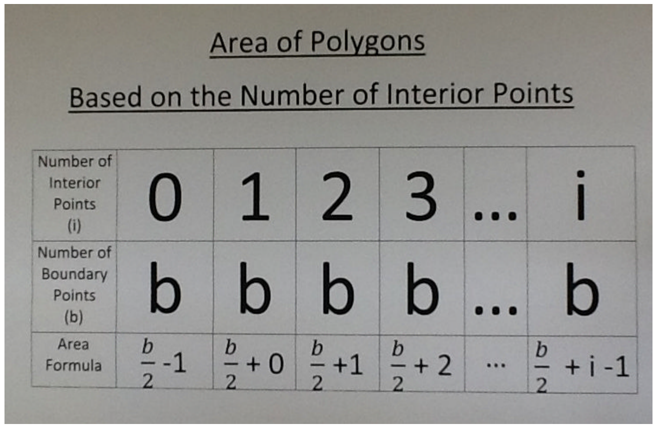


Use color with discretion—only where it helps convey a mathematical point. Using a pointer prevents you from blocking the material on your posters as you explain concepts. A two-foot wooden dowel, available at any home improvement store, is perfect for a pointer. Telescoping pointers and laser pointers are available in office supply stores and on line. They are easier to transport than longer wooden dowels. Posters can be transported in extra large zipper-lick plastic bags, which are sold in all supermarkets. The pointer will actually fit in the bag, along with your binder.



What should you put on your posters? Certainly, any diagrams you intend to refer to should be put on a poster. Re-creating them on the blackboard wastes time and doesn't produce neat results. Geometric concepts always benefit from visual aids. Tables that display patterns should be visible to your audience. As you create your posters, try to imagine how difficult it would be to *hear* the development without *seeing* it. Most likely, the speech would be verbally cumbersome, and its essence could elude the audience.

You can add material to your posters by using adhesive-backed Velcro. Put printed matter you want to paste up during your presentation on a small piece of foam board, and back it with Velcro. Then you can add it to your poster as part of the development. For example, the formulas in the last line of this poster could have been pasted up using Velcro as they were developed. Olivia's poster on Pick's Theorem is shown here.


Area of Polygons

Based on the Number of Interior Points

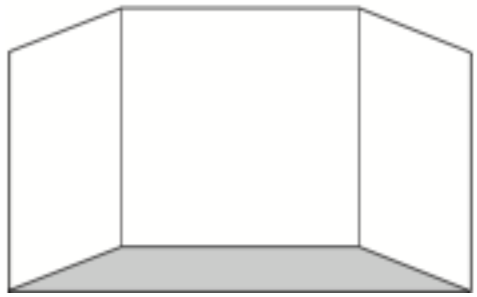
Number of Interior Points (i)	0	1	2	3	...	i
Number of Boundary Points (b)	b	b	b	b	...	b
Area Formula	$\frac{b}{2}-1$	$\frac{b}{2}+0$	$\frac{b}{2}+1$	$\frac{b}{2}+2$	...	$\frac{b}{2}+i-1$

You can purchase a pad of disposable dry-erase whiteboard sheets at educational supply stores and office supply stores. The sheets statically cling to the blackboard, so you can incorporate them into a chalkboard presentation. They can be cut and glued to parts of your posters that you would like to be able to erase. If you draw on the material with permanent markers, those markings will not erase. You can combine these types of markers to customize a poster that you can "work on" during your presentation. Use your imagination and creativity to expand on the poster suggestions in this section—but be careful not to let your artistic sense detract from the mathematical message of the poster.

Tri-fold displays. For some contests, you will use a display table rather than the front of a classroom. The judge will walk right up to your table, so there is no need for posters with large type. In these cases, you can make a tri-fold display. Tri-fold boards can be purchased in any office supply store, or on line. It can feature actual diagrams, tables, and text from your paper. The display should be free-standing, and it should give you as much room to display material as possible. Before you construct a display, you need to plan exactly what will be pasted onto it. Then you can take measurements and start construction.

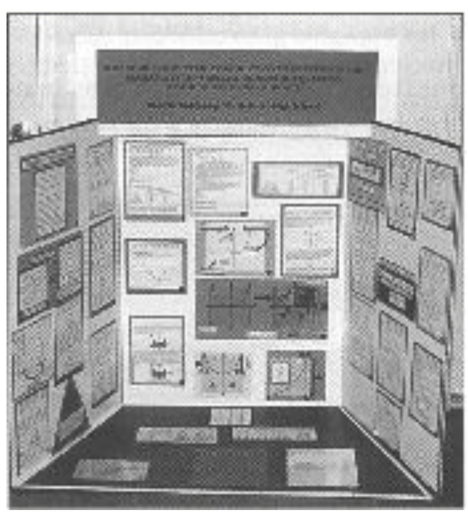
If you want a tri-fold size that is larger than the ones sold in stores, you can make your own out of foam core boards. Schematics for making one type of display out of foam-core are shown below. You will need a razor blade and a ruler. Separate the board into three folded sections. Create a "hinge" in the foam-core by taping it together on the back, with clear plastic packing tape.

You have full use of the trapezoidal space formed by the display and the table. The shaded trapezoid on the bottom gives additional space for written material. You can cut out part of the middle section to leave room for a laptop screen if a computer is part of your tri fold display. You can also include the full-length paper as part of the trapezoid. Cut out a piece of foam-core to fit in the space at the base of the display, as shown.



So far, the design has four faces you can use to display material about your research. The title can appear on a fifth face. You can fit a rectangular headpiece onto the display by slitting the sides of the display and inserting another rectangular piece of foam core in the slits. The title of your presentation goes on the headpiece. The resulting display is sturdy as well as aesthetically pleasing.

You can paste up pages of your research paper, diagrams, enlargements, reductions, overlays, and any other visuals that would enhance your presentation and that can be mounted on the display. Carefully plan the allotment of space on your display. Use your imagination to extend and alter this display description so that it meets the needs of your presentation. Danielle's completed display is shown here.



The tri fold display does not have as much room as a PowerPoint presentation would, so you need to choose what you want to display wisely. You can summarize crucial results, conjectures, and mathematical developments to make customized paste-ups for your display.

Overhead transparencies. PowerPoint presentations require a projector, and a laptop computer. These are much more expensive than the old reliable overhead projector. Your school probably has some overhead projectors. If you are making a presentation to a large group of people who are seated too far away for a display or posters to be effective, you can rely on transparencies, an overhead projector, and a screen. Overhead projectors can enlarge transparencies so they can be read from the back of an auditorium or gymnasium. The ability to handle large rooms is not the only benefit of using transparencies, however. Transparencies are equally effective in classrooms. Think of an overhead projector as a PowerPoint presentation that has no animation. Any facet of your paper can be put on a transparency using a photocopier, inkjet printer, or laser printer. Office supply stores can also make transparencies. You can make slides in color. You can make slides of photographs. Creating transparencies is much like creating PowerPoint slides.

If you plan to use many transparencies in your presentation, number them in a footer, three-hole punch them, and place them in a three-ring binder. Behind each transparency should be the printed on paper page with the same material. You can annotate this "backing sheet" with some helpful quick short notes you can use as cues during your presentation. With a scissor, cut a slit from each hole to the left margin. This will allow you to pull out your transparencies, rather than opening and closing the rings each time you need the next transparency. This is a very efficient system.

Each transparency must have a title. Some may have a conclusion at the bottom, so you are reminded to make sure the audience got the message before you go on to the next transparency. If an important development takes several transparencies, make the last line of each transparency the same as the first line of the subsequent transparency. This way, when you remove the transparency, there is a transition tot eh next one.

If you plan to generate transparencies on your computer, first make an electronic duplicate of your research paper and save the copy under the name "Transparencies." You can now use this file to compose your transparencies. You will need to enlarge the font; 12-point type is too small for a transparency. If you need to put a table or figure on a transparency, you can delete the text around the table or figure, center it, and title the transparency. You can also add any text you want. Hours of work that would normally be done by hand can be saved by making transparencies in this manner.

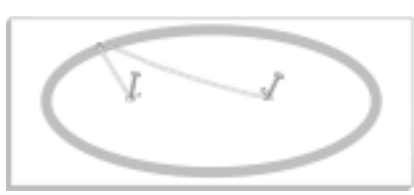
When making transparencies of algebraic developments, you can remove the left-justified reasons and merely print the centered algebraic results. Then, in your presentation, you can supply the reasons. You can take material directly from your paper and reproduce it on a transparency. The diagrams from your paper will then appear on the transparency exactly as they appear in your paper. Once such a transparency is made, you can write on it with erasable overhead pens to highlight material during your presentation. Color transparencies can be made on a laser or inkjet printer; you just need to purchase the correct product at an office supply store, or on line. For most applications, black print is sufficient. Transparencies of graph grids can be created if your presentation involves "live" graphing. This allows you to graph during your presentation. If you have a series of graduated tables or diagrams in your paper, you can develop them using transparencies. You can put the final version on a poster so it can be viewed and discussed.

Be careful not to read your transparencies verbatim. Try not to read extensive algebraic expressions often; instead, point to them and say "this expression." If you are not writing on your transparencies, stand next to the screen and use the pointer to point to the screen. When you point on the actual transparency, your hand often blocks material that the audience needs to see. Many presentations use posters *and* transparencies to combine the advantages of both.

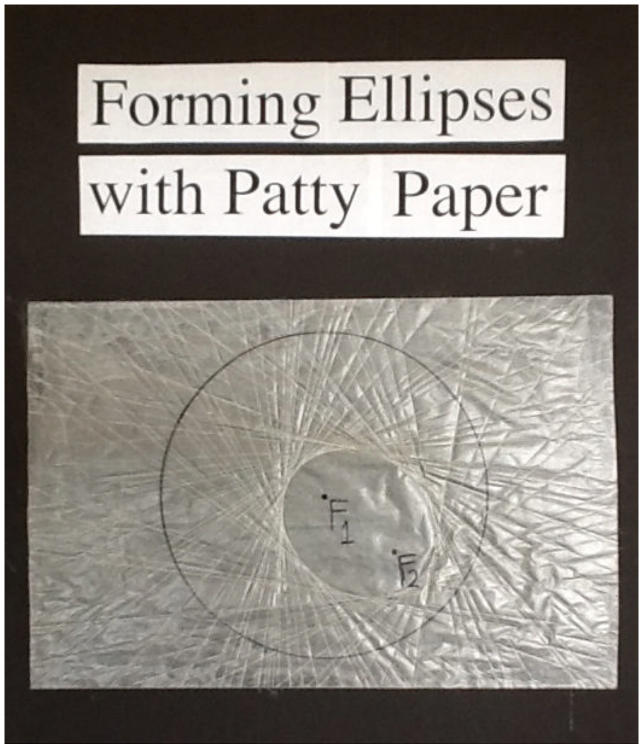
Manipulatives. When you rely on the physical handling of objects to illustrate a mathematical idea, you are using a manipulative. (Measuring instruments such as the protractor, compass, and ruler are not considered manipulatives.) If you ever used any physical hand-held aid to explore a mathematics concept, you used a manipulative—the appearance and physical manipulation of the object help teach a concept. The cutouts you make for a felt board are manipulatives. You can make manipulatives out of a wide variety of materials—clay, cardboard, Velcro, elastic, transparencies, string, styrofoam, paper cups, popsicle sticks, wood, tin cans, plastic soft drink bottles, and so on. The only limit is your imagination. If you create a manipulative to enhance your mathematical presentation, you might want to allow your audience to try it, as part of your teaser, if this is practical.

Let's look at a simple manipulative that can be used to illustrate the mathematical definition of an ellipse. An ellipse is the set of points, the sum of whose distances from two given, fixed points is a constant. You will need foam core, two push pins, string, and a marker. Push the two push pins onto the foam core board. Tie one length of string to both nails. The string should be longer than the distance between the two fixed points, as shown here.

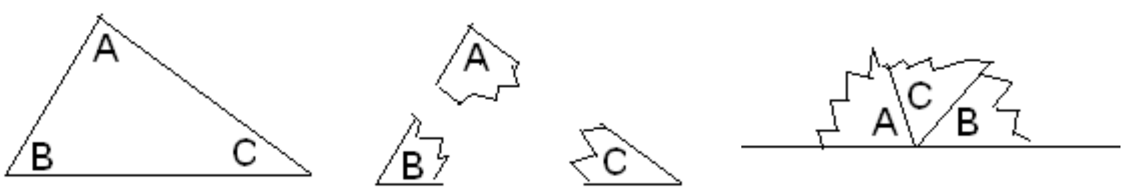
Take the marker and pull the string taut. Drag the marker around, keeping the string taut at all times. The taut string has constant length, so the sum of the two line segments is constant. As you drag the marker, you will see the ellipse take its characteristic oval shape, and your audience will better understand why an ellipse is more than just an "oval."



Cathryn created an ellipse using a large piece of wax paper by following folding instructions she read in her research. She pasted the result onto a piece of black foam core for contrast. She also had her audience create their own ellipses using wax paper as part of her teaser!

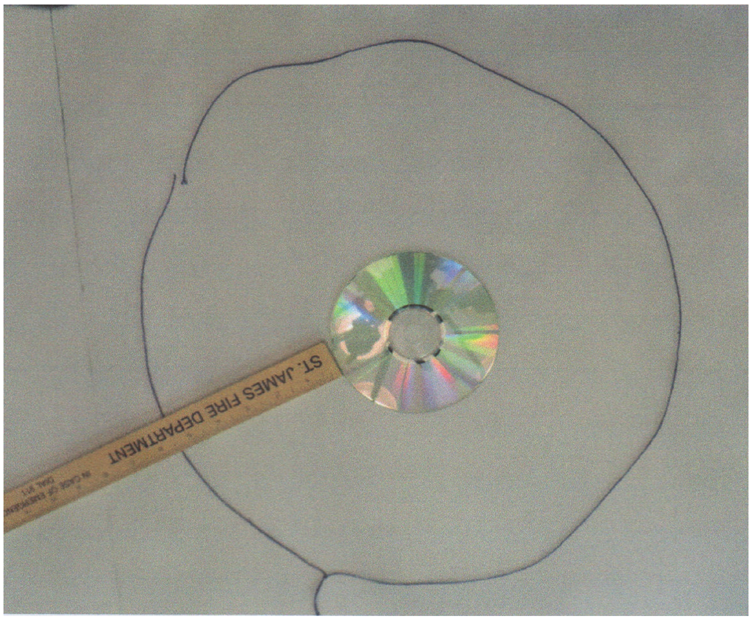


Let's use a manipulative to show that the sum of the angles of a triangle is 180 degrees. You will need scissors and a straightedge. Draw any triangle. If your audience participates, each member can draw a triangle. Label the angles *A, B,* and *C,* writing the letters *inside* the angle as shown. Cut out the triangle carefully. "Rip" the angles off at the lines shown below. Keep the three "angles" and discard the rest of the triangle.



Notice that when the three vertices are arranged around one point, the outside rays form a straight line! This is not a proof, but the fact that it occurs in all of the trials by the audience leads to a conjecture and is the motivation to attempt a proof.

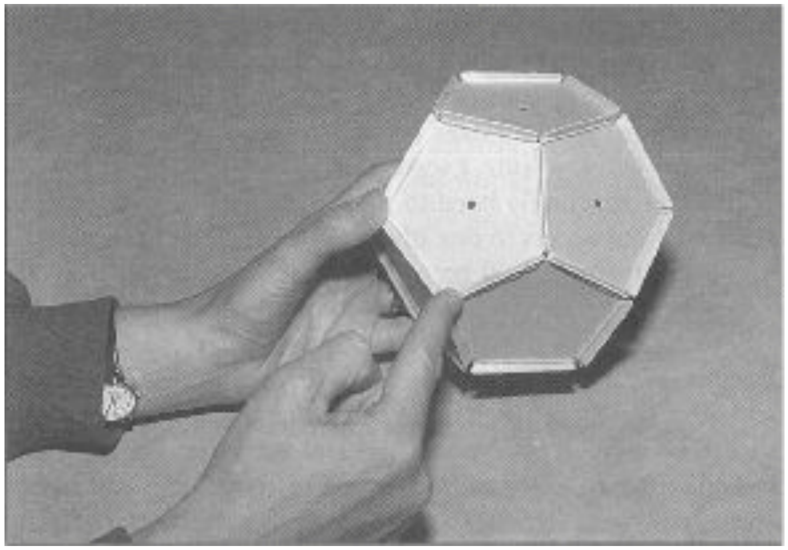
For other manipulative applications, you can take digital pictures of the results and include them in your paper and your presentation. For example, Stephen was doing research on the circumference problem introduced in Chapter 2. He had to measure the circumference of a CD using string, add 36 inches more string, and set up the string to be concentric with the CD so he could measure the gap between the CD and the string. He took a digital photo and included it in his paper and his PowerPoint presentation.



Many geometric properties lend themselves to paper folding, cutouts, and overlays. As you do your research, always think about ways manipulatives can aid your explorations, paper, and your presentation. Anything that helps your own comprehension will help your audience understand your presentation.

Models. If you are doing research on three-dimensional geometric objects, you may conduct your research and represent your findings using diagrams on paper. Perspective drawings may be sufficient for your explanations. Sometimes, however, these drawings are difficult to create, and even when they are done well they may be too confusing to illustrate a point. Let's look at two examples.

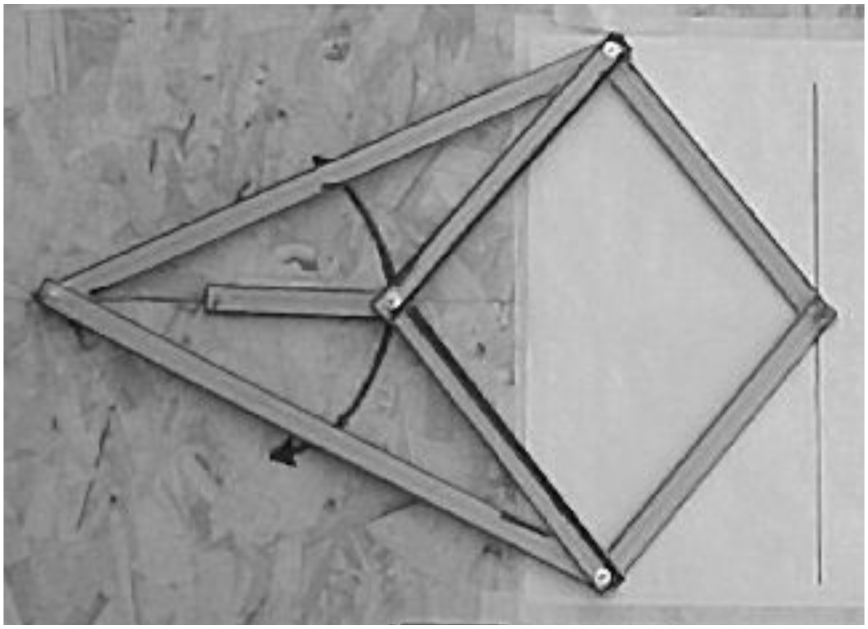
Eddie was doing research on polyhedra. He needed to research the number of faces, edges, and vertices the dodecahedron has. He had some theories, but he wanted to conclusively test his conjectures. He constructed a dodecahedron from a prefabricated manipulative kit. His work is shown here.



This manipulative allowed Eddie to prove his conjecture directly. For more intricate polyhedra, models become a necessity, especially when you are trying to convey your point in a short presentation. You can always take videos of your work with any model, and include the video in a PowerPoint presentation if you cannot present it "live."

Gino was doing research on applications of conic sections to satellite antennas. He needed to demonstrate how a cone and a plane intersect to form a parabola and a hyperbola. Gino stuffed a funnel with clay to form the cones he needed. He was then able to use a plastic knife to simulate the cutting of the cone by the plane. The different cross sections created by the different cuts allowed Gino to display the curves and show how they were formed. The model did a better job than the perspective drawings of the same cuts. Making a model is the three-dimensional version of the "draw a diagram" problem-solving strategy.

Gabrielle did a paper on inversions in the circle. She researched reflections in circular mirrors, and how a circle's reflection could be a straight lines. Related articles and Internet searches led her to a study of linkages. She built the Peaucellier Linkage shown here to test the theory.



Seeing a circular motion create a straight line was a real eye-opener, and the model really brought her presentation to a fascinating level.

Computers. Much modern mathematics research involves the use of computers. You can generate tables, search for patterns, use graphing software, or create an original program to perform a function integral to your research. The computer's ability to store and print data makes it a valuable tool for forming conjectures.

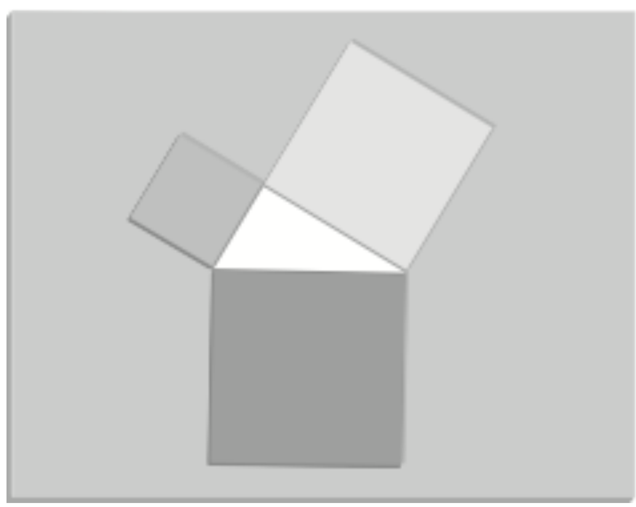
In many cases, you can print your programs and sketches and the text of your paper and reproduce them onto a transparency or poster for your presentation. Other times, it may be necessary to demonstrate the use of the computer "live" during your presentation. When using 3-dimensional computer graphing software, it is often helpful to be able it display it live. You can also copy a computer animation, make it a video, and paste it into a PowerPoint presentation.

Graphing calculator emulators. The graphing calculator is capable of much more than drawing and analyzing graphs. If you have applied the graphing calculator to your research, you might report the procedures and findings in your paper and reproduce them on a poster or transparency. If you want to use the graphing calculator during your presentation, you will need access to software that displays the calculator window.

Decide whether you absolutely need to have the calculator at your presentation. In the past, students without access to the software have free software that can copy the calculator screen and allow it to be pasted up into any document. These graphics can be merged into your paper, used to make transparencies, or included in a PowerPoint presentation.

Video. Video can be used if you need to show motion in your presentation. You can take video using a digital video camera or even your cell phone. The files can be uploaded into your PowerPoint presentation. You can write a script and record a narration as a voiceover to accompany the video.

Felt boards. Felt boards are useful in presentations that involve geometry. If, during the development, it is advantageous to be able to move shapes around on the plane, a felt board is perfect. A felt board is easy to make. Decide on the size you will need. You can probably find an inexpensive piece of scrap wood at a lumberyard; 1/2-inch plywood or particle board is excellent. Go to a fabric or crafts store and purchase enough solid-colored felt or flannel to completely cover the wood and have about 6 inches of extra fabric on each side. Stretch the felt over the board and staple it to the back of the board. Once you have chosen thebackground color,purchase contrastingcolors of felt to makethe shapes you need tomanipulate. Trace ontothe felt and cut out theshapes with scissors.Place the shapes on the felt board—notice that they “stick.” During your presentation, the felt board can be placed on a blackboard ledge. Above is a sample felt board.



Once your entire presentation is created, you need to start rehearsing it. Remember you are trying to *sell* the topic—your enthusiasm for it and your knowledge of it.

Practicing Your Oral Presentation

After you have planned your presentation and prepared your materials, it is time to practice your presentation. You need to coordinate your outline, the approximate time you will spend on each item, and the materials needed for each item. Create a Coordination of Presentation sheet to organize everything. When it has been completed, tested, and revised, print this sheet in large type on a large sheet of paper. It can be placed on the desk during your presentation, next to a watch or clock. At a glance, you will be able to see an overview of your entire presentation. The numbers under "Outline" refer to your presentation outline, which should appear in large type next to this coordination sheet. "P" refers to poster, and "PPT" to PowerPoint slides.

Notice that "Running Time" is kept as a cumulative total so you can compare your pace to your timepiece. Your cell phone's stopwatch works ideally with this column. As you rehearse, keep track of where you need more practice. When the presentation is polished, fill in the Running Time column with the approximate times you should be at each juncture.

Coordination of Presentation for a 20-Minute Presentation

OUTLINE	PAGES IN PAPER	MATERIALS	RUNNING TIME (minutes)
I A	1-3	PPT 1-6	2
I B	3-7, 11	PPT 7-11, P1	4
II A	8-14	PPT 12-15	8.5
II B	15-18	PPT 16-17, P2	11
II C	19-22	PPT 18	14.5
III A	22-24	P3	17
III B	25	Blackboard	18
IV A	26-29	PPT 19-21	20

Do not make the final, large-type copy of the Coordination of Presentation document until you have worked out all of the kinks in your talk.

Even with rehearsal, your time may not work out exactly as you planned. You do not want to rush or stretch material to fit in your allotted slot, so you may need to place contingency topics in the second half of your outline. Insert topics that you will add if there is extra time, and note topics that you will delete if you are running behind. Keep an eye on your outline's running time for the first half of your presentation, and make mental notes on what you may need to adjust.

How can you make your presentation smooth and reduce nervousness? Know your audience. The presentation must be geared to their level of mathematics and their familiarity with your research. (Possibly your paperwas submitted weeks in advance and the judges have read it.) Use segues, or transition techniques, to bridge the different parts of your paper so the presentation is seamless. One technique is to ask a question and then answer it. Be empathetic—remember that your audience hasn't studied the topic over the past few months as you have. You may have internalized some of the material so well that it is second nature to you. Pause. Give the audience time to digest what you have just said. Practice often, without an audience in the beginning.

When you are polished, you can have someone take a video of your presentation. If you are using presentation software, you can actually record a timed narration of the presentation with all slide transitions and animations in sync to your narration. You can listen back to this and critique your own presentation!

Watch the video. Athletes, politicians, teachers, and actors do this, and you, too, can use tapes to improve your delivery. Listen to the audio without watching the video. Is your communication precise, correct, and mathematically sophisticated? Are you speaking in full, well-defined sentences? Do you sound enthusiastic about the topic?

Can you “learn” to sound enthusiastic? Yes, you can. There are two TV personalities worth discussing in reference to enthusiasm. You can do an Internet search to view and hear their styles. One is Billy Mays, a spokesman for cleaning products. Look at one of his commercials. He gets so :into” his cleaning product—you can hear it in his voice. Aim for that style—you’ll probably fall short of it and that should be just right for your math paper. Another TV celebrity is Jerry Seinfeld. Search for some footage from his TV shows and listen to the inflection in his voice. Shoot for that, and you’ll most likely fall short and have improved inflection.

Once you are well rehearsed, begin “dress rehearsals” with audiences. Parents, friends, and siblings may not understand the research itself, but they will help you simulate the conditions of performing in front of a live audience. If you plan to enter a judged contest, rehearse by having your instructor watch your presentation and make a list of positive points and aspects that need work. Make presentations to your math class, the math department or math club, a school faculty meeting, and so on. Always remain receptive to constructive criticism. Field questions from your practice audiences. Try to anticipate questions the judges might ask, and prepare the answers. Solid preparation is your best defense against nervousness. Show enthusiasm, speak clearly, and pace yourself. Remember to pause, and don’t rush. Smile, look members of your audience in the eye, and use inflection. Letting your voice rise and fall in pitch and volume indicates your interest and excitement, and it will help your audience get involved. Think of inflection as verbal italics, and use it to add emphasis and emotion to your presentation. Then your passion for your topic will come through. You should be very confident once you have ironed out all the wrinkles in your presentation.

In Chapter 13 there is a rubric judges can use to rate presentations. You can use this as a guide to rate the video of your presentation.

On the day of your presentation, be ready. Dress appropriately—this presentation is a reflection on you, your school or college, your respect for your work, and your respect for the contest and the judges. Nervousness is good if it made you rehearse, revise, and critique your presentation. However, on the day of your presentation, nervousness is counterproductive, so try and leave your nervousness at home that day.

You will need to make a checklist of all the supplies you need for your presentation.

Check this carefully before you leave home, so you are sure you have everything. Your checklist might include the following:

- A tabbed, annotated copy of your paper
- A binder with print outs of your visuals
- Posters
- A stopwatch
- A pointer
- Transparencies, markers, and overhead projector
- Laptop and projector and all necessary cords
- Manipulative materials
- Copies of your teaser to hand out

Many schools require students to give an exhibition of a major project as a requirement of graduation. This exhibition can take many forms. Consult your instructor and guidance counselor to find out whether your research and presentation in mathematicscan qualify as your exhibition. As you review the steps you've followed over the chapters in this book, note carefully ways in which your research satisfies your school's requirements for performance assessments and exhibitions.

And, most of all, remember to *sell* and not just *tell*!

Chapter 11
Sample Pages from Actual Papers

A Sample of a Student's Mathematics Research Paper

The following mathematics research paper showcases some of the possibilities in the world of mathematics research. Dylan Smith wrote the paper as an eleventh grade student at North Shore High School. Dylan picked out a topic from a mathematics journal. The article he chose was from the November 1992 issue of *Mathematics and Informatics Quarterly*. It was a 3-page article entitled "Tangent Circles in Isosceles Triangles," and it was written by Hiroshi Okumura.

Mathematics journals have articles that are several pages long on selected math topics. Dylan's 70+ pages (condensed to 20+ pages for *Writing Math Research Papers*) are the result of the work he did based on reading a 3-page article! Often these articles are condensed to save space, so Dylan had to fill in the algebraic and geometric gaps in the article. He then extended some of the ideas in the article, and prepared his paper in the spirit of the annotation projects. As a result, his paper does a much more comprehensive job of covering the material, and it is easier to understand than the original article. It also contains findings that the original article did not include.

Selected Single Pages from Actual Student Math Research Papers

Dylan's paper was lengthy due to his new explorations and findings. The length of the paper is never an issue. There are papers that might be 15 pages and some that are continued over more than one school year which are 150 pages! The number of pages is not the important issue—working to your potential is. Seeing how you can push yourself to produce professional-looking academic material has benefits far beyond the mathematics that is learned, because these skills apply to all endeavors in your career and life, as well as your schooling.

The single page cross sections of student papers allow you to use different usages, formatting, styles, figures, captions, table titles, and other nuances of research papers. You may get some ideas by looking at these samples.

Presented at the Long Island Math Fair, Spring 2013

AN EXPLORATION OF THE RATIO OF RADII OF SETS OF PAIRWISE TANGENT CIRCLES IN ISOSCELES TRIANGLES

Dylan Smith
Grade 11
North Shore High School

ABSTRACT

This paper is an exploration of the ratio of radii of sets of pairwise tangent circles in isosceles triangles. The length of the common external tangent to two circles is found numerically and algebraically. It is then used to help solve a problem involving pairwise tangent circles in isosceles triangles. Through this problem, various ratios between the different radii of these circles are obtained. A calculator program is written in order to obtain these ratios. Once the first 26 are found, an unsuccessful attempt is made to use regression functions to predict the following ratios. Some recommendations for further research are to find a regularity that would allow the prediction of upcoming ratios without solving them by hand or by the use of a computer program.

PROBLEM STATEMENT

We will look at a problem involving tangents and circles. A **tangent** is a line that touches a circle at just one point in one plane. The first type of tangent we will look at is common external tangents. The **line of centers** is an imaginary line that extends from the center of one circle to another. A **common external tangent** is one that exists without an intersection of the line of centers. A common external tangent is shown below with a dotted-line to represent the line of centers.

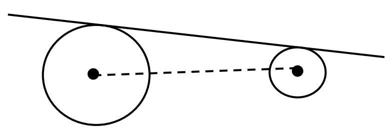


Figure 1. The common external tangent does not intersect the line of centers.

The second type of tangent we will look at is common internal tangents. A **common internal tangent** is one that exists with an intersection of the line of centers.

A common internal tangent is shown below with a dotted line to represent the line of centers.

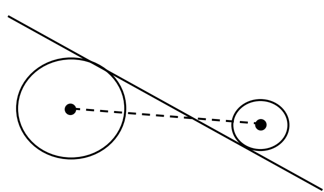


Figure 2. The common internal tangent intersects the line of centers.

Next, we will discuss the relationship of the location of circles to the number of tangents as well as the length of those tangents. Shown below are several examples of circles with various distances from one another and their respective tangents. The dots that are on some of the following **tangent lines** are meant to distinguish the **tangent segments**. Tangent segments are only the segment of the tangents extending from a point on one circle to the corresponding point on another. A tangent line is one that is an extension of tangent segments and continues forever.

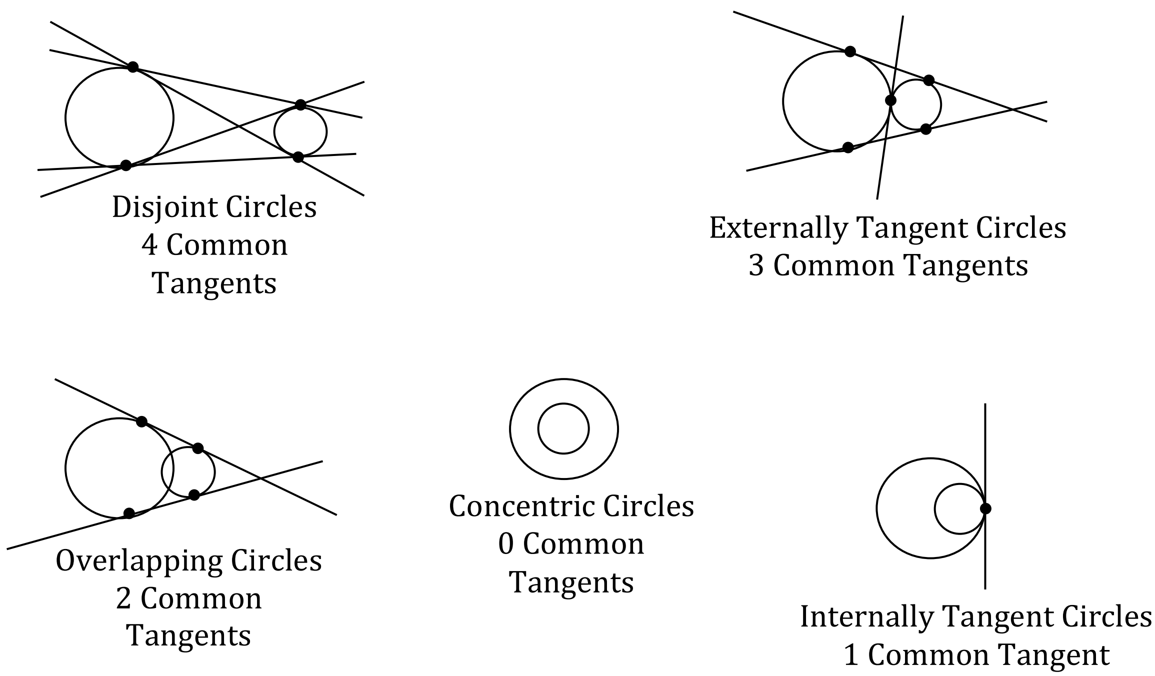


Figure 3. Various examples of circles and their tangents.

Throughout this paper we will be discussing the distance between circles. The distance between two circles will be defined as the distance from the two points closest to each other. These two points will be on the line of centers, as shown.

Figure 4. Examples of what constitutes the distance between circles.

- What is the relationship between the distance between two circles and the length of their tangents?
- How can this relationship be used to find the ratio of the radii of selected circles in isosceles triangles?

RELATED RESEARCH

We will start by looking at a geometric problem involving tangents. The following diagram shows circles P and L externally tangent to each other at M. $\overline{GN}$ is a common tangent where G and N are the points of tangency. To solve for the length of GN, which is what the problem asks, we can draw LS perpendicular to PG. First, we will label our diagram with variables.

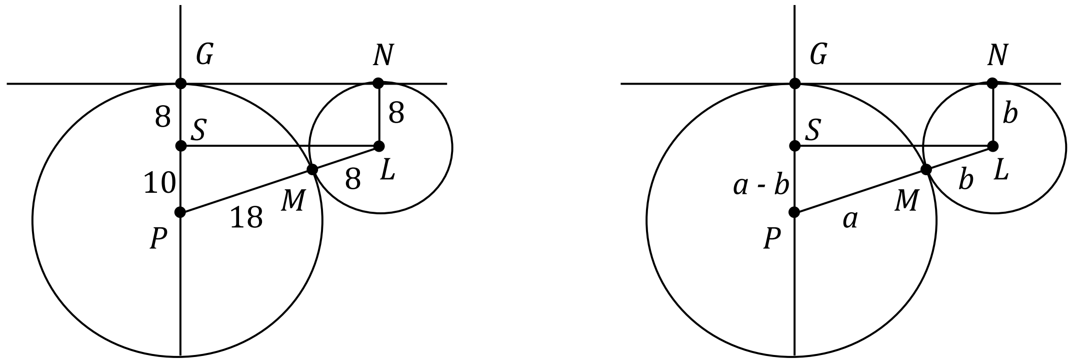


Fig. 5. Setting up the algebraic solution.

We can assume

$$a = \text{the radius of circle } P$$

We can also assume

$$b = \text{the radius of circle } L$$

LN is a radius of circle *L*, therefore

$$LN = b$$

LN and *SG* are perpendicular to the same parallel lines, therefore

$$SG = LN$$

$$SG = b$$

Also, PG is a radius of circle P

$$PG = a$$

$$PG - SG = a - b$$

We can see from the diagram that

$$PG - SG = PS$$

$$PS = a - b$$

Also we can see from the diagram that

$$ML = a + b$$

Now let's plug our variables into the Pythagorean theorem using ΔPSL.

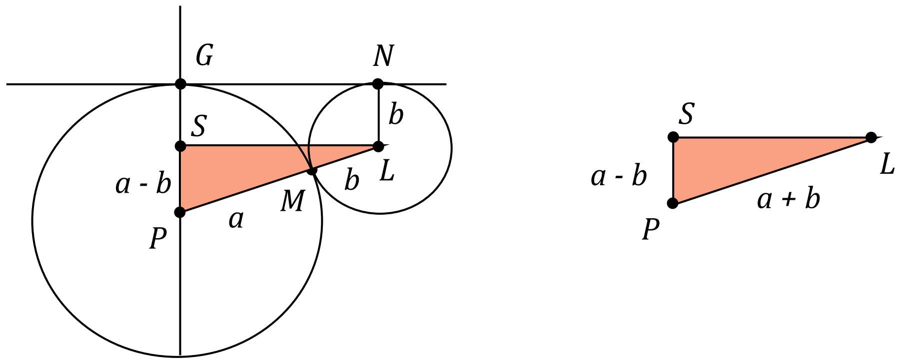


Figure 6. ΔPSL is reprinted with the corresponding variables for each side.

Rewrite the Pythagorean Theorem.

$$(a+b)^2 = (a-b)^2 + (SL)^2$$

Multiply.

$$a^2 + 2ab + b^2 = a^2 - 2ab + b^2 + (SL)^2$$

Add *2ab* to both sides.

$$a^2 + 4ab + b^2 = a^2 + b^2 + (SL)^2$$

Subtract $a^2 + b^2$ from both sides.

$$4ab = (SL)^2$$

Take the square root of both sides.

$$2\sqrt{ab} = SL$$

We can conclude:

$$
\begin{array}{l}
2\sqrt{ab} = SL \\
SL = GN \\
\hline
\therefore GN = 2\sqrt{ab}
\end{array}
$$

The common external tangent to two circles is twice the square root of the product of the radii. This fact will be used to help with the solution to a problem about tangent circles in isosceles triangles.

Tangent Circles in Isosceles Triangles

In Figure 7, the radius of circle *O* is *R* and the radii of the smaller congruent circles are *r*. We are going to find the ratio of the radius of the larger circle, R, to the radius of one of the congruent smaller circles, *r*. We can let r = 1.

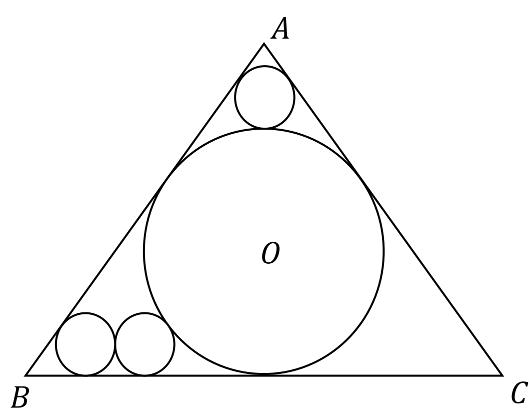


Fig. 7. Isosceles triangle ABC with tangent circles.

We will now draw a new diagram with a new triangle within isosceles ΔABC. To construct this next diagram we will let *X* and *Y* be the centers of the two of the smaller circles, as shown. We will connect these centers to form the hypotenuse. Next, we will draw a line from *Y* parallel to *BC* that is cut off by a perpendicular line drawn from *X*. This will form right ΔXYZ. Now let's look at Figure 7.

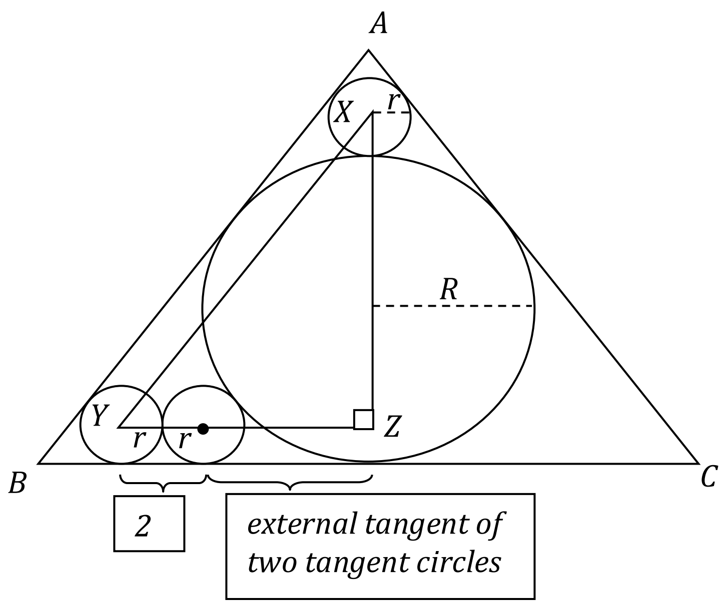


Figure 7. The dotted lines represent radii of their corresponding circles.

From the Pythagorean theorem we can conclude that

$$XY^2 = YZ^2 + XZ^2 \qquad \text{(Equation 1)}$$

Also we have proved previously that

$$\textit{external tangent of two tangent circles} = 2\sqrt{\textit{product of radii}}$$

We know that *YZ* is made up of *2r* and an external tangent of two tangent circles.

$$YZ = 2r + 2\sqrt{Rr} \qquad \text{(Equation 2)}$$

To explain the next part of our solution, we must return to our diagram and add some additional points.

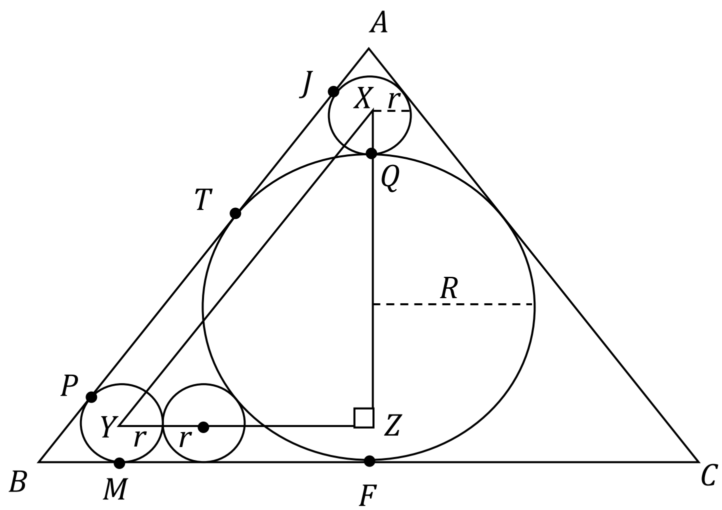


Figure 8. Points *P*, *T*, *M*, *Q*, and *F* have been added as various tangent points.

Tangents to the same circle from the same point are congruent. *BP* and *BM* are tangents to the same circle from the same point.

$$BP \cong BM$$

BT and *BF* are also tangents to the same circle from the same point and thus congruent.

$$BT \cong BF$$

From Figure 8, we can see that *PT* and *MF* are sections of tangents *BT* and *BF* respectively.

$$BT - BP = PT$$
$$BF - BM = MF$$

We can conclude that

$$BT - BP \cong BF - BM$$

$$PT \cong MF$$

$$MF \cong YZ$$

$$PT \cong YZ$$

TJ is an external tangent of two tangent circles, which we have proved to be equal to $2\sqrt{product\ of\ radii}$.

$$TJ = 2\sqrt{Rr}$$

Thus we can conclude that

$$XY = YZ + 2\sqrt{Rr}$$

We can substitute for *YZ* based on the information from Equation 3.

$$XY = 2r + 2\sqrt{Rr} + 2\sqrt{Rr}$$

We see that *XQ* is a radius of circle *X*.

$$XQ = r$$

The distance from *F* to *Z* is also equal to the length of a radius of circle *Y*. This is because *YZ* and *MF* are parallel lines that intersect the center of circle *Y* and are tangent to the circle, respectively. This means that any two corresponding points on those two

lines are the distance of that circle's radius.

$$FZ = r$$

Therefore, we can shift line *XZ* down so that it starts at *Q* and ends at *F*. In doing this we can be sure that the distance of the line will not be affected. The shift is displayed below.

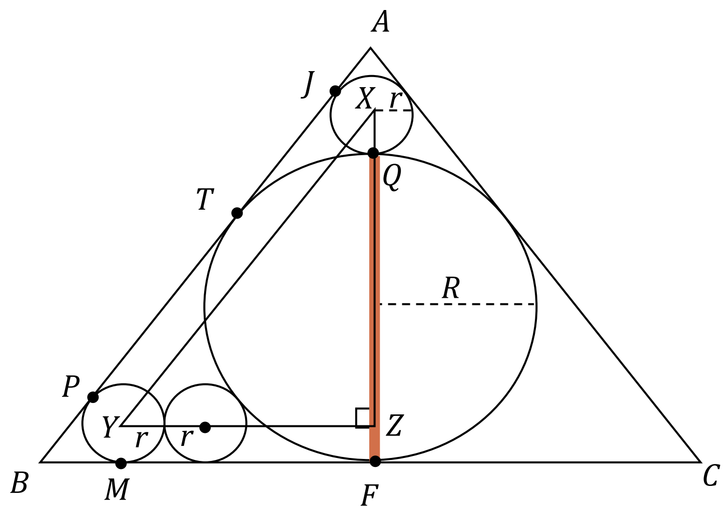


Figure 9. The red line indicates line *XZ* after the shift.

The red line is the diameter of circle *T* and thus is equal to the length of the sum of two of its radii.

$$XZ = 2R$$

We know recall our information from Equation 2 and substitute our values.

$$YZ^2 + XZ^2 = XY^2$$

$$(2R)^2 + \left(2r + 2\sqrt{Rr}\right)^2 = \left(2r + 2\sqrt{Rr} + 2\sqrt{Rr}\right)^2$$

Rewrite the equation to see each term that is squared.

$$(2R)(2R) + \left(2r + 2\sqrt{Rr}\right)\left(2r + 2\sqrt{R}\right) = \left(2 + 2\sqrt{R} + 2\sqrt{R}\right)\left(2 + 2\sqrt{R} + 2\sqrt{R}\right)$$

Multiply $(2R)(2R)$.

$$4R^2 + \left(2 + 2\sqrt{R}\right)\left(2 + 2\sqrt{R}\right) = \left(2 + 2\sqrt{R} + 2\sqrt{R}\right)\left(2 + 2\sqrt{R} + 2\sqrt{R}\right)$$

Multiply $\left(2 + 2\sqrt{R}\right)\left(2 + 2\sqrt{R}\right)$.

$$4R^2 + 4 + 4\sqrt{R} + 4\sqrt{R} + 4R = \left(2 + 2\sqrt{R} + 2\sqrt{R}\right)\left(2 + 2\sqrt{R} + 2\sqrt{R}\right)$$

Multiply $\left(2 + 2\sqrt{R} + 2\sqrt{R}\right)\left(2 + 2\sqrt{R} + 2\sqrt{R}\right)$

$$4R^2 + 4 + 4\sqrt{R} + 4\sqrt{R} + 4R = 4 + 8\sqrt{R} + 8\sqrt{R} + 16R$$

Add $4\sqrt{R} + 4\sqrt{R}$

$$4R^2 + 4 + 8\sqrt{R} + 4R = 4 + 8\sqrt{R} + 8\sqrt{R} + 16R$$

Add $8\sqrt{R} + 8\sqrt{R}$.

$$4R^2 + 4 + 8\sqrt{R} + 4R = 4 + 16\sqrt{R} + 16R$$

Subtract 4 from both sides.

$$4R^2 + 8\sqrt{R} + 4R = 16\sqrt{R} + 16R$$

Subtract $8\sqrt{R}$ from both sides.

$$4R^2 + 4R = 8\sqrt{R} + 16R$$

Subtract $4Rr$ from both sides.

$$4R^2 = 8\sqrt{R} + 12R$$

Subtract $8r\sqrt{Rr}$ from both sides.

$$4R^2 - 8\sqrt{R} = 12R$$

Subtract $12Rr$ from both sides.

$$4R^2 - 12R - 8\sqrt{R} = 0$$

Divide both sides by 4.

$$R^2 - 3R - 2\sqrt{R} = 0$$

Factor out a $\sqrt{R}$.

$$\sqrt{R}\left(R\sqrt{R} - 3\sqrt{R} - 2\right) = 0$$

Divide both sides by $\sqrt{R}$. Usually, it is not allowed to divide by a variable, but that is because we would be eliminating the chance that the variable could equal 0. In this case, *R* = the radius of a circle and therefore cannot be 0.

$$R\sqrt{R} - 3\sqrt{R} - 2 = 0$$

Rewrite $-3\sqrt{R}$ as $\sqrt{R} - 4\sqrt{R}$ to facilitate factoring.

$$\underline{R\sqrt{R} + \sqrt{R}} \underline{- 4\sqrt{R} - 2} = 0$$

The reason we rewrite $-3\sqrt{R}$ as $\sqrt{R} - 4\sqrt{R}$ is to allow for factoring by grouping, for we can see that the red terms have a $\sqrt{R}$ in common and the two blue terms have a -2 in common. However, our facilitating is not complete because when we try to actually factor by grouping we got the following result:

$$\underline{\sqrt{R}\left(R + 1\right)} \underline{- 2\left(\sqrt{R} + 1\right)}$$

Unfortunately, the terms inside the parentheses do not match and so our factoring by grouping is not successful, indicating we must go back to our original equation (before we tried to factor) and add something else to facilitate the factoring so that it works. The right most parenthesis only consist of a $\sqrt{R}$, but needs to be a *R* to match the other set of parentheses. Let's go back to our original and try again.

Add 0 in the form of $2R-2R$ to facilitate factoring.

$$R\sqrt{R}+2R+\sqrt{R}-2R-4\sqrt{R}-2=0$$

We see that when we tried to factor by grouping and failed, the red terms needed a $\sqrt{R}$ term and the blue terms were missing a *R* term. To manufacture a *R* term for the blue terms, we added *2R*. The reason we add *2R* and not just *R* is because it must be have -2 as a factor so that it fits into our factoring by grouping method. We must also add a *-2R* to the equation, though, because we cannot just add numbers without reason. By adding *2R -2R* we are really adding 0, which is allowed. Coincidentally, the *-2R* used as a way to add 0 actually manufacturers a $\sqrt{R}$ term for the red terms, and therefore completing our facilitating of factoring by grouping.

Factor out $\left(\sqrt{R}-2\right)$

$$\left(\sqrt{R}-2\right)\left(R+2\sqrt{R}+1\right)=0$$

Factor the trinomial.

$$\left(\sqrt{R}-2\right)\left(\sqrt{R}+1\right)\left(\sqrt{R}+1\right)=0$$

Simplify.

$$\left(\sqrt{R}-2\right)\left(\sqrt{R}+1\right)^2=0$$

Set the two cases equal to 0 in order to solve for *R*.

$$\left(\sqrt{R}-2\right)=0$$

Add 2 to both sides.

$$\sqrt{R}=2$$

Square both sides.

$$R=4$$

$$\left(\sqrt{R}+1\right)^2=0$$

Find the square root of both sides.

$$\left(\sqrt{R}+1\right)=0$$

Subtract 1 from both sides.

$$\sqrt{R}=-1$$

Square both sides.

We can reject the solution that states that *R* = 1 because it was given the circle *O* was larger than the congruent smaller circles which have the radius *r* = 1. Therefore we know that *R = 4* is the correct solution.

$$R=4$$

In addition, we recall that we made *r* = 1 and therefore can now set up our ratio between the radius of circle *O* to the radii of the congruent smaller circles.

$$R:r=4:1$$

Circle *O* has a radius four times the size to that of each of the smaller congruent circles. We have found a solution to our problem.

If we continue the logical steps used in the previous cases, we can find the R:r ratio for isosceles triangles with different numbers of smaller circles in the lower left base angle region.

<u>Table 1. Ratio of Circle Radii</u>

Number of Small Circles (in the bottom left corner)	Equation	*R : r*
1	n / a	3 : 1
2	$R\sqrt{R}-3\sqrt{R}-2=0$	4 : 1
3	$R\sqrt{R}-3\sqrt{R}-4=0$	4.8216402 : 1
4	$R\sqrt{R}-3\sqrt{R}-6=0$	5.5474447 : 1
5	$R\sqrt{R}-3\sqrt{R}-8=0$	6.21023 : 1
6	$R\sqrt{R}-3\sqrt{R}-10=0$	6.827183 : 1
7	$R\sqrt{R}-3\sqrt{R}-10=0$	7.4086976 : 1
8	$R\sqrt{R}-3\sqrt{R}-14=0$	7.9616532 : 1
9	$R\sqrt{R}-3\sqrt{R}-16=0$	8.490896 : 1
10	$R\sqrt{R}-3\sqrt{R}-18=0$	9 : 1
11	$R\sqrt{R}-3\sqrt{R}-20=0$	9.491695 : 1
12	$R\sqrt{R}-3\sqrt{R}-22=0$	9.9681251 : 1
13	$R\sqrt{R}-3\sqrt{R}-24=0$	10.431013 : 1
14	$R\sqrt{R}-3\sqrt{R}-26=0$	10.881768 : 1
15	$R\sqrt{R}-3\sqrt{R}-28=0$	11.321562 : 1
16	$R\sqrt{R}-3\sqrt{R}-30=0$	11.751384 : 1
17	$R\sqrt{R}-3\sqrt{R}-32=0$	12.172076 : 1
18	$R\sqrt{R}-3\sqrt{R}-34=0$	12.584364 : 1
19	$R\sqrt{R}-3\sqrt{R}-36=0$	12.988878 : 1
20	$R\sqrt{R}-3\sqrt{R}-38=0$	13.38617 : 1
21	$R\sqrt{R}-3\sqrt{R}-40=0$	13.776728 : 1
22	$R\sqrt{R}-3\sqrt{R}-42=0$	14.160986 : 1
23	$R\sqrt{R}-3\sqrt{R}-44=0$	14.539328 : 1
24	$R\sqrt{R}-3\sqrt{R}-46=0$	14.921202 : 1
25	$R\sqrt{R}-3\sqrt{R}-48=0$	15.279621 : 1
26	$R\sqrt{R}-3\sqrt{R}-50=0$	15.642167 : 1

If we look at the middle column of Table 1, we see that the equations are all very similar. The only part that seems to be changing is the *A* value. We will now try to find a relationship between the numbers of small circles in the bottom left corner of our problem diagram (the leftmost column in Table 9) and the changing *A* values. We see that the *A* value can be divided by 2 and then have 1 added to it. If we do this backwards we obtain the general formula for this pattern.

$$A = 2(c-1)$$

Where: *A = the A value*
c = the number of small circles in the bottom left corner

Since our general equation for this problem is $R\sqrt{R} - 3\sqrt{R} - A = 0$, we can substitute for *A*.

$$R\sqrt{R} - 3\sqrt{R} - 2(c-1) = 0$$

Now we have a general solution that allows us to find the ratio that our problem asks for, the ratio of the radius of the larger circle to that of the congruent smaller circles, by just plugging in numbers.

Integer values for *R* seem to be fairly rare. In order to find only these integer values for *R* without calculating by hand, we can write a program for a calculator. The calculator used to write the program is a TI-84 plus from Texas Instruments™.

On the following page there is a screen shot of the original program written as well as explanations of the functions of each line as it corresponds to our problem. In short, the program solves the equation $R\sqrt{R} - 3\sqrt{R} - A = 0$ for *R* where the value of *A* is increasing by 2 and the *R* value can only be an integer.

EXPLANATIONS

```
PROGRAM:SMITH
:For(R,4,300,1)
```

This line indicates the values we are "solving" for. In actuality, the calculator will be plugging these numbers into our equation (line 3) to see if, when plugged in, will satisfy the equation. The line literally means make *R* equal to all numbers in the interval of 4-300, while increasing by increments of 1. First the calculator will make *R* = 4.

```
:For(A,2,20000,2
)
```

This line indicates the variable that is changing (but keep in mind we really have two variables changing). The line literally means make *A* equal to all numbers in the interval of 2-20000, while increasing by increments of 2. First the calculator will make *A* = 2.

```
:If R√(R)-3√(R)-
A=0
```

This line represents the equation that we will be trying to "solve". This is the start of an If/Then/Else statement, and if the *R* value, in this case 4, makes this equation true, with its corresponding *A* value, in this case 2, the program will move onto the THEN statement. If the *R* value does NOT make this equation true, with its corresponding *A* value, the program will move onto the ELSE statement. Based on whether or not the corresponding *R* and *A* values satisfy the equation, there are two possible outcomes.

```
:Then
```

Here we have the THEN line, which is immediately followed by what the calculator must do if the equation is true with the corresponding *R* and *A* values, in this case 4 and 2.

```
:Disp A,R
```

If the equation is true with the corresponding *R* and *A* values, THEN the calculator will display the values for each variable on the home screen.

```
:Else
```

Here we have the ELSE line, which is immediately followed by what the calculator must do if the equation is NOT true with the corresponding *R* and *A* values, in this case 4 and 2.

```
:End
```

This END is the ending of the If/Then/Else subroutine. This line acts as the end of the first of three subroutines.

```
:End
```

This END is the ending of the "For A" loop. The calculator will now return to the 2nd line and repeat the following lines of the program, but with the second value being used to represent A, in this case it is 4.

```
:End
```

This END is the ending of the "For R" loop. The calculator will now return to the 1st line and repeat the following lines of the program, but with the second value being used to represent R, in this case it is 5.

When we run this program we obtain the following values.

Table 2. Program Values

Value of A in $R\sqrt{R}-3\sqrt{R}-A=0$	*R*
2	4
18	9
52	16
110	25
198	36
332	49
488	64
702	81
970	100
1298	121
1692	144
2158	169
2702	196
3330	225
4048	256
4862	289
5778	324
6802	361
7940	400
9198	441
10582	484
12098	529
13752	576
15550	625
17498	676
19602	729

We notice that all of the values for *R* that satisfy this equation, with a corresponding *A* value, are perfect squares. In fact, the entire column is made up of increasing consecutive perfect squares. When we look at the *A* values, however, it is hard to see an obvious pattern. In order to find a common pattern among these *A* values, we will use various regression functions on a calculator. If we find a pattern, we will be able to predict the next *A* values without having to check.

When we use our calculator to find the different types of regression with the *A* and *R* values from Table 2, we must look for a common coefficient of 1. To test the regressions, we must first insert our values into *List* 1 and *List 2* of the calculator.

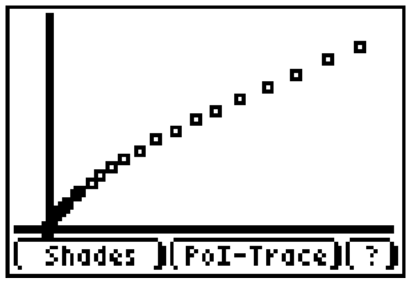


Figure 10. The scatter plot graphs the values from Table 10 as *(x , y)* coordinates.

Next, we will try different types of regressions. The different types we will try are linear regression, quadratic regression, cubic regression, quartic regression, logarithmic regression, exponential regression, and power regression, in that order. The following screenshots are the results of the regressions.

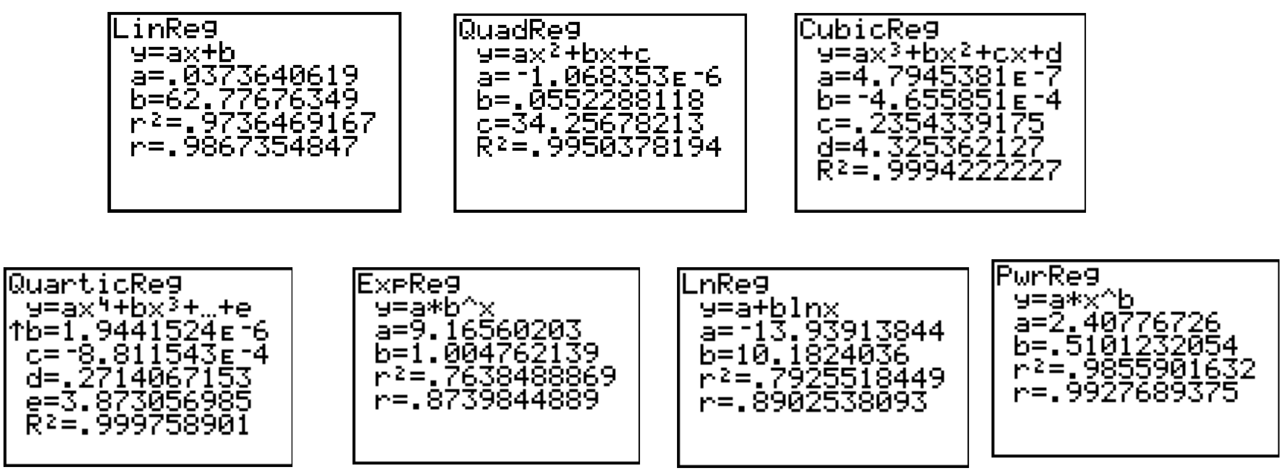


Figure 11. None of the regressions give an equation that satisfies all of the points.

Unfortunately, through the use of regressions, there is no equation that satisfies all of the points. The common coefficient is very close to 1 because the equation is close to satisfying all of the points given, but does not do it fully. Thus, we have not yet found a concrete formula relationship between the R and A variables.

RECOMMENDATIONS FOR FURTHER RESEARCH

We have previously done cases where R has often been an irrational number. Now we are going to explore the cases when R is rational. Further research should include finding a way to determine for what cases R going to be a rational number.

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Selected Single Pages from Actual Student Math Research Papers

PROBLEM STATEMENT

The area of a triangle formula is

$$A = \frac{1}{2}bh.$$

A **perfect triangle** is a triangle with natural number sides in which the area is numerically equivalent to its perimeter. If we were to find the area of a right triangle, the two legs would represent the base and the height. We are looking for a perfect triangle.

Table 1. Looking for Perfect Triangles in Pythagorean Triples

Sides	Area (A)	Perimeter (P)	Does A=P?
5, 12, 13	30	30	Yes
6, 8, 10	20	24	No
8, 15, 17	120	40	No
9, 40, 41	180	90	No
7, 24, 25	84	56	No

What if you are not given the base and height, but only the sides? In this case, you can use Heron's formula. It is a formula that finds the area of a triangle with the three side lengths. In a triangle inequality, the sum of the two smaller sides has to be greater than the largest side in order to be a triangle. First, you have to find half the perimeter, or s.

$$s = \frac{a+b+c}{2}$$

Once you have s, you can then calculate the area of the triangle using Heron's formula. Let a, b, and c represent the three sides in a triangle.

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

This represents the area of *one* of the triangles. It must be multiplied by 4 to get the value for the area of the rhombus.

$$A = (4)\left(\frac{1}{8}\right)d_1 d_2$$

$$A = \frac{1}{2}d_1 d_2$$

We can use this formula to find the area of a rhombus.

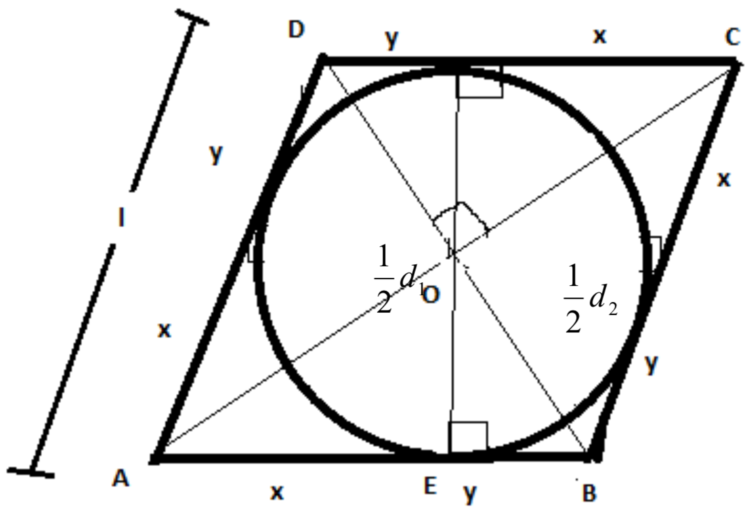


Figure 10. A circle inscribed in a rhombus.

We are able to label multiple values of x and y because when 2 tangent segments are drawn from the same external point, they are congruent.

The area of the rhombus can be represented by

$$A = \frac{1}{2}d_1 d_2$$

Multiply by 1 in the form of $2 \cdot \frac{1}{2}$

$$A = 2\left(\frac{1}{2}d_1\right)\left(\frac{1}{2}d_2\right)$$

Non-Linear Diophantine Equations

Let's look at using modular arithmetic to solve non-linear Diophantine equations.

$$x^2 - 15y^2 = -1$$

If we use mod 4 on this equation we get

$$x^2 + y^2 = 3$$

Now we have 2 perfect squares adding up to 3. Let's look at the perfect squares in mod 4.

Table 1. Perfect Squares (mod 4)

Perfect Square	Perfect Square (Mod 4)
1	1
4	0
9	1
16	0
25	1
36	0
49	1
64	0
81	1
10	0
121	1

Notice that in Table 1, when a perfect square is expressed in mod 4, the only possibilities for the congruencies are 0 or 1. This can help us solve if there are solutions to an equation. Using the example from above, we see that

$$x^2 + y^2 = 3.$$

We know that the two squared terms can only be equal to 1 or 0, so let's look at the possibilities for this equation, This means that the value of *a* is any point higher than the absolute maximum of this graph. To find the absolute max of the graph in Figure 18, we would click 2nd, then Calc. Then we would use the fourth option, maximum. We scroll the blinking cursor to the left side of the highest point of the graph and click Enter. Next, we scroll the blinking cursor to the right of the highest point of the graph and click Enter twice. This is how we find the maximum of the graph.

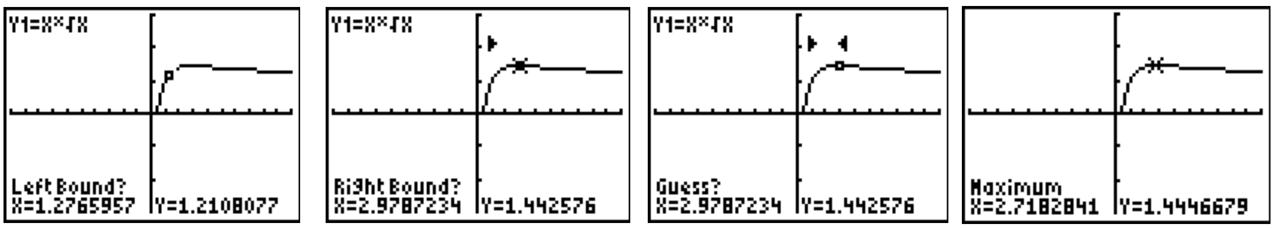


Figure 19. Finding the maximum of y = $\sqrt[x]{x}$.

The maximum of this graph is (2.7182841, 1.4446679). The x is the same value as e and the y is the same we derived algebraically.

In the previous figures, figures 13 – 15, we saw a similar number when comparing the y-coordinate of the maximum and the value we were deriving for a in y = a^x.

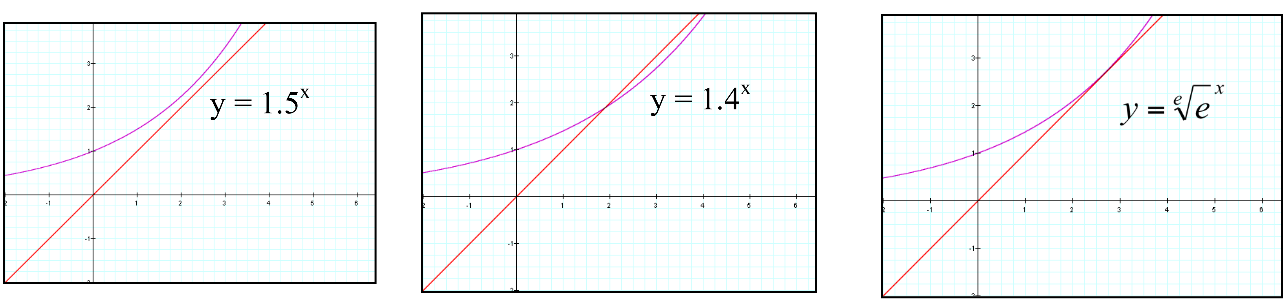


Figure 21. Comparing our trial and error method to our proof of x < a^x.

Finding Patterns in Pythagorean Triples

Now that we found equations that will find Pythagorean Triples, we will look for patterns in the triples. First, we will let

$$k = j + 1.$$

Table 13. Finding Patterns in Triples

k = j + 1	j	x	y	z
3	2	5	12	13
5	4	9	40	41
9	8	17	144	145
6	5	11	60	61

Table 13 shows that another way to find x is to add 1 to j to get k, and then add k and j together. In this table we observed that if k is one more than j, z is one more than y. Now to see if this will be true every time we have to prove the pattern. We substitute j + 1 for k into our three equations to find Pythagorean Triples.

$$k = j + 1$$

We begin by substituting j + 1 for k into our first equation.

$$x = k^2 - j^2$$

$$x = (j + 1)^2 - j^2$$

Square.

$$x = (j + 1)(j + 1) - j^2$$

Multiply.

$$x = j^2 + j + j + 1 - j^2$$

$$y^2 = \left(1 - \frac{(x-a)^2}{a^2}\right)(2a-1)$$

Get a common denominator in the first set of parentheses so we can make it one fraction.

$$y^2 = \left(\frac{a^2}{a^2} - \frac{(x-a)^2}{a^2}\right)(2a-1)$$

Combine the fractions, multiply out $(x - a)^2$, and distribute the negative sign.

$$y^2 = \left(\frac{a^2 - x^2 + 2ax - a^2}{a^2}\right)\left(\frac{2a-1}{1}\right)$$

Simplify the numerator in the first set of parentheses.

$$y^2 = \left(\frac{-x^2 + 2ax}{a^2}\right)\left(\frac{2a-1}{1}\right)$$

Split the a^2 in the denominator evenly between the two fractions. Then, split the numerator in the first set of parentheses.

$$y^2 = \left(\frac{-x^2}{a} + \frac{2ax}{a}\right)\left(\frac{2a-1}{a}\right)$$

Commute the fractions in the first set of parentheses.

$$y^2 = \left(\frac{2ax}{a} - \frac{x^2}{a}\right)\left(\frac{2a-1}{a}\right)$$

Cancel out the *a*'s in the first fraction in the first set of parentheses.

$$y^2 = \left(2x - \frac{x^2}{a}\right)\left(\frac{2a-1}{a}\right)$$

Factor out the greatest common factor.

Counting Numbers	Triangular Numbers	Tetrahedral Numbers	Sums of the First n Tetrahedral Numbers
$a_n = n$	$a_n = \frac{1}{2}\left(n^2 + n\right)$	$a_n = \frac{1}{6}\left(n^3 + 3n^2 + 2n\right)$	$a_n = \frac{1}{24}\left(n^4 + 6n^3 + 11n^2 + 6n\right)$

Next factor the parts of the equations inside the parentheses.

Counting Numbers	Triangular Numbers	Tetrahedral Numbers	Sum of the First n Tetrahedral Numbers
$a_n = \frac{n}{1}$	$a_n = \frac{(n+0)(n+1)}{2}$	$a_n = \frac{(n+0)(n+1)(n+2)}{6}$	$a_n = \frac{(n+0)(n+1)(n+2)(n+3)}{24}$

Now the pattern in the numerator is very apparent. If we consider the counting numbers to be sum 0 then the numerator is (n + 0)(n + 1)…(n + k) where k is the number of summations. Now let's look just at the denominator.

Table 5. Denominators of Consecutive Sums of the First n Counting Numbers

Number of Summations	Denominator
0	1
1	2
2	6
3	24

Notice that to generate the denominator for sum 1, multiply the denominator of sum 0 by 2. Then to generate the denominator for sum 2, multiply the denominator of sum 1 by 3, and then to generate the denominator for sum 3, multiply the denominator of sum 2 by 4. This means the denominator is 1 × 2 × 3 × … (k + 1), which is the same as (k + 1)! Based on the pattern we can form the conjecture that the n^{th} term of k^{th} summation is

$$a_n = \frac{(n+0)(n+1)(n+2)(n+3)\ldots(n+k)}{(k+1)!}.$$

Combining Descartes' and Hudde's Methods

Recall that Descartes' method of taking a derivative using a tangent circle is tedious when applied to functions with a degree larger than 2. By applying Hudde's Rule to Descartes' Circle Condition, we will be able to take the derivative of a function more easily than we previously could, even for functions of a higher degree. We begin with the general equation of a power function $y = x^n$ and the general equation of a circle with radius r and center at $(v,0)$:

$$(v-x)^2 + y^2 = r^2.$$

As shown in Figure 8, n can be odd or even.

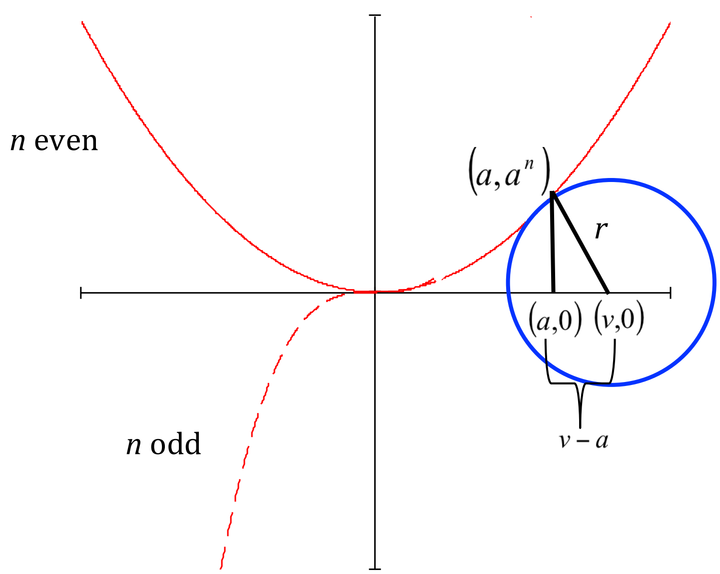


Fig.8. The function $y = x^n$ where n is any real number.

We know $y = x^n$, so we will substitute x^n for y in the equation of our circle.

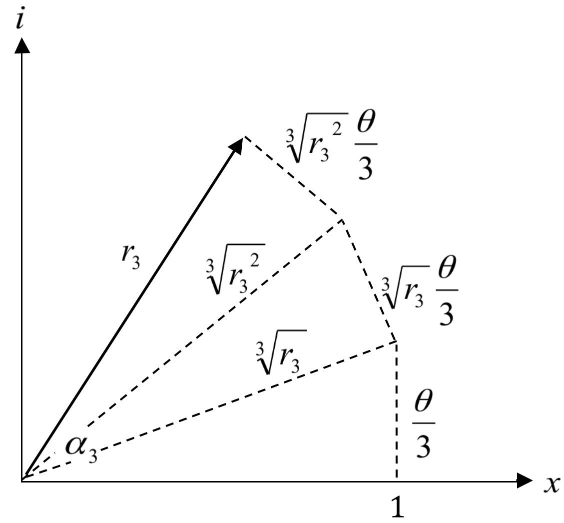


Fig. 14. $z_3 = \left(1 + \frac{i\theta}{3}\right)^3$

We notice an interesting pattern when graphing our vectors. This pattern is exhibited in Table 2.

<u>Table 2. Vectors and their Corresponding Angles</u>

Vector	Length of vector	Angle with the *x*-axis
$z_1 = (1 + i\theta)^1$	$r_1 = \sqrt{1+\theta^2}$	$\alpha_1 = \tan^{-1}(\theta)$
$z_2 = \left(1 + \frac{i\theta}{2}\right)^2$	$r_2 = 1 + \frac{\theta^2}{4}$	$\alpha_2 = 2\tan^{-1}\left(\frac{\theta}{2}\right)$
$z_3 = \left(1 + \frac{i\theta}{3}\right)^3$	$r_3 = \left(\sqrt{1+\frac{\theta^2}{9}}\right)^3$	$\alpha_3 = 3\tan^{-1}\left(\frac{\theta}{3}\right)$

We can use this information to stipulate equations to find z_n, r_n, and α_n. We already know the equation for z_n from page 22.

$$z_n = \left(1 + \frac{i\theta}{n}\right)^n$$

In Figure 3, there is a square cut diagonally into triangles and then rearranged into another triangle. The formula for the area of a triangle is base X height divided by 2. The area of the triangle stays the same as the square. Both are 64 square units.

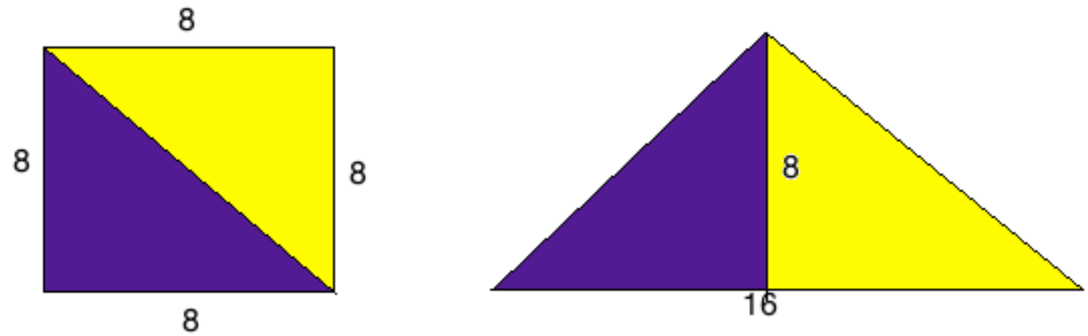


Figure 3. This is an 8 by 8 square turned into a triangle with base 16 and height 8.

In Figure 4, there is an 8 by 8 square and a 5 by 13 rectangle. These different polygons are right triangles and trapezoids. The red polygon is a right triangle, just like the blue polygon. The green polygon is a trapezoid as is the yellow polygon. The polygons were taken from the square and were rearranged into a rectangle. The weird thing about the rearrangement is that the rectangle's area is different than the square's area! We call this a Fibonacci square and a Fibonacci rectangle.

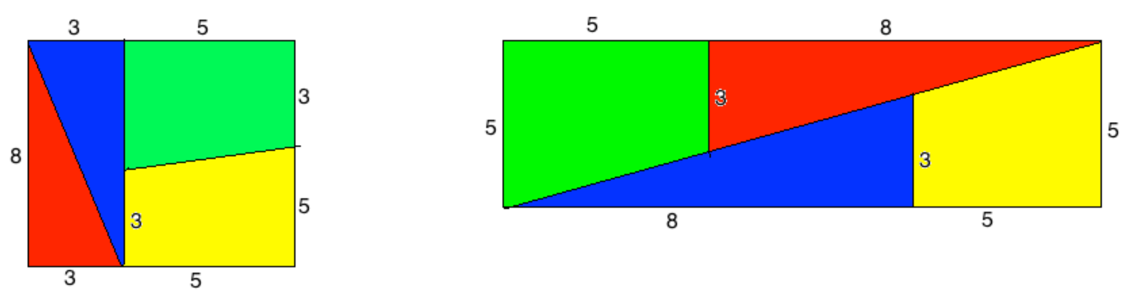


Figure 4. The area of the square is 64 , while the area of the rectangle is 65.

Chapter 12

Resources

In Chapter 2, you read about finding a topic. There are many resources you can use to find a topic or to locate additional material for a topic you have already started. Chapter 12 lists dozens of possible resources under the following categories:

- periodicals/journal articles
- problem solving publications
- research topics from books
- Internet sources

These categories are not mutually exclusive. For example, a book on problem solving may inspire a research topic or provide an extension to your research topic. All books and articles in all the categories are potential sources of material for research papers. Standard textbooks are not listed here. In cases where you need supplementary material from a textbook, see your instructor or the math department chairperson. The periodicals section also includes journal articles that students have used in their research.

Finding an Article in a Journal

Below is a list of popular journals that have articles you can use in your research.

- *CMC Communicator*
- *College Mathematics Journal*
- *COMAP Consortium*
- *Duodecimal Bulletin*
- *Eisenhower National Clearinghouse Focus*
- *The Fibonacci Quarterly*
- *Focus on Learning Problems in Mathematics*
- *Games Magazine*
- *Journal of Mathematical Behavior*
- *Mathematics and Informatics Quarterly*
- *Mathematics Magazine,*
- *The American Mathematical Monthly*
- *Math Horizons*

- *The Mathematics Teacher*
- *Journal for Research in Mathematics Education*
- *Mathematics Teaching*
- *The Pentagon—A Mathematics Magazine for Students*
- *Quantum*
- *Plus Magazine*
- *School Science and Mathematics*
- *Scientific American*
- *TI Cares*

You may find some of these publications cited in your bibliography or in the bibliography of one of your readings. Look in the periodicals section of your school library, college library, or public library. If you are interested in a particular back issue or subscription information or have other questions, do an Internet search to find the current contact information for the organization that publishes the periodical. Many of these publishers have copies of back issues as electronic files that can be e-mailed to you, or downloaded from their website. In some cases you may have to be a member of an organization, and in some cases there might be a fee to obtain the article. Comb the websites and look for links that lead you to the contact information you need. Keep in mind that some journals go out of print and may not be currently publishing. You will have to creatively use Internet searching to track down these publications. A librarian may be able to help you too.

College libraries also have some of these journals, or may know how they can be accessed. Some periodicals have an annual index that you might find useful. Additionally, every state has a mathematics teacher organization that has a publication. An Internet search will help you find the organization's website for your state. Ask a librarian if you need help tracking down a journal or a specific article. If you plan to contact the publisher for information, make sure you do this immediately. You must allow time for a response, and you don't want to delay your research work unnecessarily. The Internet has expedited the work of searching tremendously!

Chapter 2 acquainted you with some advantages of using a journal article to find a topic.

- The articles provide a narrow, focused topic appropriate for a research paper.
- Thousands of students have used these articles with success.
- The articles are well written and carefully edited.

The articles supply the "skeleton"—a frame for your research. You can supply the "meat"—your extensions of the article's ideas—in a logical, guided fashion. This section will give you information on finding such an article. Keep in mind that articles from older issues of journals can be just as useful as articles from current or recent journals. Older articles also allow you to apply modern technology—programming, computer software, graphing calculators—to look at the topic in a modern light. Students have done excellent papers using articles from the National Council of Teachers of Mathematics journal for secondary educators, the *Mathematics Teacher*. Use contact information on the website www.nctm.org if you need to get a copy of an article. They are available to non-NCTM members for a small fee. You need the name of the article, the author, and the date of the issue the article was in. Find out if your teacher is an NCTM member—this may give free access to the articles.

While most articles are appropriate, there are some excellent ones that are perfect for high school students. The following pages list dozens of articles on which students have based research papers. The articles are from *Mathematics Teacher, Mathematics and Informatics Quarterly, Math Horizons, The Pentagon, Quantum*, and *High School Mathematics and its Applications* (HiMAP) Project. You may want to suggest to your school or local library that they order some of these journals. Additionally, you might want to advertise in your state's math teachers association print and online publications to get hard copies of old journals from teachers who are retiring and would like to donate them.

These articles are listed in date order with the most recent date listed first. Articles from any decade are excellent; do not infer that a more article recent is better. Student have done articles from the 1930s and created superb papers! One interesting aspect of articles that pre-date calculators and computers is that there are new ways to tackle the problems using technology, and that opens up many avenues for original exploration. These new methods complement the "old school style" very eloquently!

Mathematics Teacher

May 2014	Technology-Enhanced Discovery
April 2014	The Circle Approach to Trigonometry
March 2014	A Rationale for Irrationals
February 2014	Cultivating Deductive Thinking with Angle Chasing
January 2014	Angry Birds Mathematics: Parabolas and Vectors
October 2013	Geometry of the Fibonacci Matrix
September 2013	Are All Infinities Created Equal?
May 2013	Derivative of Area Equals Perimeter-Coincidence or Rule?
January 2013	Gaming: The Law of Large Numbers
November 2012	A Road for Every Wheel
March 2012	Exploring Conics: Why Does $B^2 - 4AC$ Matter?
November 2011	The Shape of an Ellipse
October 2011	Ellipses and Orbits: An Exploration of Eccentricity
September 2011	Investigating Zeros of Cubics with GeoGebra
August 2011	Delving into Limits of Sequences
May 2011	Back to Treasure Island
September 2010	Calculating Pythagorean Triples
August 2009	How Sample Size Affects a Sampling Distribution
August 2009	An Intriguing Exponential Inequality
February 2009	Slicing a Cube

August 2008	Generating Problems from Problems and Solutions from Solutions
May 2008	Heron Triangles and Moduli Spaces
February 2007	Sprinklers and Amusement Parks
March 2007	What Else Can You Do with an Open Box?
August 2006	Card Folding: An Investigation with Limits
January 2006	Understanding Conic Sections Using Alternate Graph Paper
February 2005	Another Way to Divide a Line Segments into *n* Equal Parts
January 2005	Is a Triangle Determined by the Length of its Angle Bisectors?
May 2003	On Inscribed and Escribed Circles of Right Triangles
May 2003	A Direct Approach to the Sine of the Sum of Two Angles
March 2003	Paper Folding and Conic Sections
February 2003	More Meaning form the Geometric Mean
December 2002	Exploring the Four-Points-on-a-Circle Theorems
January 2002	Alternative Geometric Constructions
November 2001	Rugby and Mathematics
May 2001	Don't Be Square-A Geometric Excursion
May 2001	Dividable Triangles-What Are They?
May 2001	The Equation of a Triangle
October 2000	A Triangle Divided: Investigating Equal Areas
October 2000	Building Connections Among Polynomial Functions
September 2000	Modeling Soft Drink Packaging
May 2000	Can Euler's Line be Parallel to a Side of a Triangle?
April 2000	Discovering an Optimal Property of the Mean
March 2000	The Coefficient of Determination
October 1999	Counting Triples, Triangles, and Acute Triangles
May 1998	The Cevian Problem
April 1998	The Conic Sections in Taxicab Geometry
January 1995	Circular Graphs: Vehicles for Conic and Polar Coordinates
February 1993	If Pythagoras had a Geoboard
May 1991	A Monte Carlo Application to Approximate Pi
April 1991	Pascal' s Triangle and Fibonacci Numbers
March 1991	The Probability that a Quadratic Equation Has Real Roots
January 1991	Odd Factors and Consecutive Sums: An Interesting Relationship
April 1990	Seven Ways to Find the Area of a Trapezoid
April 1989	Pythagoras Meets Fibonacci
October 1988	A Number Game—Summing Consecutive Positive Integers
January 1988	Triangles of Equal Area and Perimeter
March 1987	Spheres in a Cone: Proving the Conic Sections
November 1986	Pythagorean Triples
January 1986	Factoring Polynomials and Fibonacci
October 1985	A Surprising Fact About Pythagorean Triples
May 1985	Measuring the Area of Golf Greens
February 1982	A Squeeze Play on Quadratic Equations
March 1981	Area = Perimeter
September 1979	On the Radii of Inscribed and Escribed Circles of Right Triangles
April 1979	Serendipity on the Area of a Triangle

December 1976	Circles, Chords, Secants, Tangents and Quadratic Equations
May 1974	Pick' s Rule
May 1974	Some Methods for Constructing the Parabola
November 1966	Geometric Solution of a Quadratic Equation
October 1966	Radii of the Appolonius Contact Circles
May 1966	Even More On Pascal's Triangle and the Powers of Eleven
April 1966	A Geometric Approach to the Conic Sections
March 1966	Pseudo-Ternary Arithmetic

Math Horizons

April 2013	Tails in High Dimensions
November 2012	Higher Fashion Meets Higher Mathematics
September 2012	An Unanticipated decimal Expansion
September 2012	Bezier Curves with a Romantic Twist
February 2012	A Dozen Proofs that 0 = 1
November 2011	Trigonometry without Triangles
September 2011	Newton's Proof of Heron's Formula
September 2009	Polishing Some Visual Gems
February 2003	Fitch Cheney' s Five Card Trick
November 2002	Is the SAT I Exam Relevant?
November 2002	You Can't Go Wrong With Triangles
April 2002	Digging For Squares
February 2002	President Garfield and the Pythagorean Theorem
April 2001	The Bridges of Konigsberg
September 2001	A Dozen Questions about the Powers of Two
September 1999	Cycloidal Areas Without Calculus

Mathematics and Informatics Quarterly

March 2002	Adventures in Area Analysis
March 2002	Geometric Constructions V
July 2000	Forgotten Theorems
September 1999	A Sangaku Theorem With Proof
June 1998	Auxiliary Elements in Problem Solving Pushing the Limits
September 1997	Three Concentric Circles
June 1997	An Investigation into Goldbach' s Conjecture
November 1994	Equations, Inequalities and Graphs
June 1994	Algebraic Relations for the Inscribed Orthodiagonal Quadrilaterals
March 1994	Expected and Unexpected Solutions
May 1993	Tangent Nine-Point Circles
November 1992	Cyclic Quadrilaterals with Integer Sides and Diagonals
September 1992	The Median Toward the Hypotenuse
May 1992	On the Malfatti Problem for the Equilateral triangle
March 1992	Orthodiagonal Quadrilaterals Again

March 1992	On the Bobillier Theorem
March 1992	Figures of Equal Area
August 1991	The Folded Square
February 1991	What is the Use of the Last Digit?
February 1991	Simple Properties of the Orthodiagonal Quadrilaterals
February 1991	The Bobillier Theorem

The Pentagon

Spring 2003	Baseball: A Statistical Analysis
Spring 2003	Transformations of the Unit Circle
Spring 2002	Foci of Conic Sections
Spring 1999	A Ratio Proof of the Pythagorean Theorem
Spring 1999	Two Proofs of the Pythagorean Theorem using Area

Quantum

July 2000	Geometric Surprises
May 2000	Fermat's Little Theorem
March 2000	About the Triangle
November 1999	The Feuerbach Theorem
March 1998	Points of Interest: Unique Locations Within a Triangle
January 1998	Constructing Quadratic Solutions
January 1998	The Lunes of Hippocrates
November 1997	Unidentical Twins
September 1997	An Ant on a Tin Can
November 1990	The Natural Logarithm

High School Mathematics and its Applications (HiMAP)

The following "modules" are sold separately, and some are available online. Check the COMAP website (www.comap.com) for new titles. You can contact COMAP for information on downloading these modules from their website, or purchasing them.

A Mathematical Look at the Calendar
A Uniform Approach to Rate and Ratio Problems
Applications of Geometrical Probability
Architecture Designpack
Businesspack
Codes Galore
Decision Making and Math Models
Drawing Pictures with One Line
Enviropack
Loads of Codes

Medipack
Sociopoliticopack
The Mathematical Theory of Elections

College libraries may have hard copies of recent journals, and possibly some dating back a few years. Call your local college library to find out if they have any of the recommended journals. Going to the library is a good step to locating an article that interests you, because you can look at many articles while you're there.

Problem Solving

Most books on problem solving are timeless. Some of the best classic problems are years, decades, even centuries old. Libraries and bookstores have many recently published titles about problem solving, but don't shy away from a book because it is old. Many problem-solving books from previous decades are still available in bookstores today or can be found in a library. Be aware that some older books may be more difficult to find.

Many problem-solving books give solutions and hints for some or all of the problems. You can use the problems to improve your problem-solving skills, to enhance your research paper, to find a topic, or for just plain fun. Use the following list, or find other books in your library. Although the following book list, in totality, contains over 1000 brainteasers, it merely scratches the surface of the available books on problem solving. The Internet can be used to expand your problem solving resources.

Artino, R., Gaglione, A., and Shell, N. *The Contest Problem Book IV: American High School Mathematics Examination 1973–1982*. Washington, D.C.: Mathematical Association of America, 1983.

Barr, S. *Mathematical Brain Benders*. New York: Dover, 1982.Bates, N., and Smith, S. *101 Puzzle Problems*. Concord, Mass.: Bates, 1980.

Bolt, B. *The Amazing Mathematical Amusement Arcade*. New York: Cambridge University Press, 1984.

Conrad, S., and Flegler, D. *The First High School Math League Problem Book*. Tenafly, N.J.: Math League Press, 1989.

Conrad, S., and Flegler, D. *The Second High School Math League Problem Book*. Tenafly, N.J.: Math League Press, 1992.

Ecker, M. *Getting Started in Problem Solving and Math Contests*. New York: Franklin-Watts, 1987.

Engel, A. *Problem Solving Strategies*. New York: Springer-Verlag, 1997.

Erickson, M., and Flowers, J. *Principles of Mathematical Problem Solving*. New Jersey: Pearson Education, 1998.

Flener, F. *Mathematics Contests: A Guide for Involving Students and Schools*. Reston, Va.: NCTM, 1990.

Gardner, M. *The Colossal Book of Mathematics: Classic Puzzles, Paradoxes, and Problems*. New York: W.W. Norton and Co., 2001.

Gardner, M. *Perplexing Puzzles and Tantalizing Teasers*. New York: Dover, 1980.

Gardner, M. *Wheels, Life and Other Mathematical Amusements*. New York: W. H. Freeman, 1985.

Gilbert, G., Krusemeyer, M., and Larson, L. *The Wohascum County Problem Book*. Washington, D.C.: Mathematical Association of America, 1993.

Greenes, C., Schulman, L., Spungin, R., Chapin, S., and Findell, C. *Mathletics—Gold Medal Problems*. Providence, R.I.: Janson Publications, 1990.

Greitzer, S. *International Math Olympiads 1959–1977*. Washington, D.C.: Mathematical Association of America, 1978.

Grosswirth, M., and Salny, A. *The Mensa Genius Quiz Book*. Reading, Mass.: Addison-Wesley, 1981.

Grosswirth, M., and Salny, A. *The Mensa Genius Quiz Book 2*. Reading, Mass.: Addison-Wesley, 1983.

Halmos, P. *Problems for Mathematicians Young and Old*. Washington, D.C.: Mathematical Association of America, 1991.

Hardy, K., and Williams, K. S. *The Green Book of Mathematical Problems*. New York: Dover Publications, 1997.

Herr, T., and Johnson, K. *Problem Solving Strategies—Crossing the River with Dogs and Other Mathematical Adventures*. Berkeley, Calif.: Key Curriculum Press, 2001.

Houghton, G. *Common Sense Puzzles*. New York: Hart, 1984.Hunter, J. *Challenging Mathematical Teasers*. New York: Dover, 1980.

Klamkin, M. *International Math Olympiads 1978–1985*. Washington, D.C.: Mathematical Association of America, 1986.

Reeves, C. *Problem Solving Techniques Helpful in Mathematics and Science*. Reston, Va.: NCTM, 1987.

Salkind, C. *The Contest Problem Book II: American High School Mathematics Examination 1961–1965*. Washington, D.C.: Mathematical Association of America, 1966.

Salkind, C. *The Contest Problem Book III: American High School Mathematics Examination 1966–1972*. Washington, D.C.: Mathematical Association of America, 1973.

Schoen, H. L., ed. *Teaching Mathematics through Problem Solving: Grades 6–12*. Reston, Va: NCTM, 2004.

Serebriakoff, V. *Puzzles, Problems, and Pastimes for the Superintelligent*. Englewood Cliffs, N.J.: Prentice-Hall, 1983.

Shushan, R. *Games Magazine Big Book of Games*. New York: Workman, 1984.

Stevenson, F. W. *Exploratory Problems in Mathematics*. Reston, Va.: NCTM, 1992.

Yawin, R. *Math Games and Number Tricks*. Middletown, Conn.: Field Publications, 1987.

Zeitz, P. *Art and Craft of Problem Solving*. New York: John Wiley and Sons, 1999.

Research Topics from Books

In Chapter 2, you learned that articles from periodicals and books can help you find a research topic. Some books have short mathematical "essays" that are excellent springboards for research papers. These essays are just like journal articles. A single book might contain dozens of essays. Some of these books are listed below. Your library will have other titles; also check with the math department chairperson and your instructor to see if they have some of these books or others like them. Online searches will invariably lead to many articles on topics related to the key descriptor words you used for your search.

Aaboe, A. *Episodes from the Early History of Mathematics.* Washington, D.C.: Mathematical Association of America, 1975.

Adler, I. *Readings in Mathematics.* Lexington, Mass.: Ginn, 1972.

Allinger, G. D., et al. *Mathematics Project Handbook, Fourth Edition.* Reston,

Va.: NCTM, 1999. Austin, J. *Applications of Secondary School Mathematics—Readings from the Mathematics Teacher.* Reston, Va.: NCTM, 1991.

Beiler, A. *Recreations in the Theory of Numbers.* New York: Dover, 1966.

Bollobas, B. *Littlewood's Miscellany.* New York: Cambridge University Press, 1986.

Bolt, B. *More Mathematical Activities.* Cambridge: Cambridge University Press, 1988.

Bowers, J., and Bowers, H. *Arithmetical Excursions.* New York: Dover, 1961.

Brase, C. H., and Brase, C. P. *Understanding Basic Statistics.* Boston: Houghton Mifflin, 1997.

Campbell, D., and Stanley, J. *Experimental and Quasi-Experimental Designs for Research.* Dallas: Houghton Mifflin, 1963.

Consortium for Mathematics and Its Applications. *High School Lessons in Mathematical Applications.* Lexington, Mass.: COMAP, 1993.

Croft, H., Falconer, K., and Guy, R. *Unsolved Problems in Geometry.* New York: Springer-Verlag, 1995.

Dalton, L., and Snyder, H. *Topics for Mathematics Clubs.* Reston, Va.: NCTM, 1990.

Department of Mathematics and Computer Science, North Carolina School of Science and Mathematics. *New Topics for Secondary School Mathematics: Geometric Probability.* Reston, Va.: NCTM, 1988.

Department of Mathematics and Computer Science, North Carolina School of Science and Mathematics. *New Topics for Secondary School Mathematics: Matrices.* Reston, Va.: NCTM, 1988.

Eccles, F. *An Introduction to Transformational Geometry.* Menlo Park, Calif.: Addison-Wesley, 1971.

Farmer, D. W. *Groups and Symmetry: A Guide to Discovering Mathematics.* Providence, R.I.: American Mathematical Society, 1996.

Farmer, D. W. and Stanford, T. B. *Knots and Surfaces: A Guide to Discovering Mathematics.* Providence, R.I.: American Mathematical Society, 1996.

Foster, J. *Data Analysis Using SPSS for Windows.* Thousand Oaks, Calif.: Sage, 2001.

Gardner, M. *Mathematical Circus.* Washington, D.C.: Mathematical Association of America, 1992.

Gardner, M. *Mathematical Magic Show.* Washington, D.C.: Mathematical Association of America, 1990.

Garland, T. *Fascinating Fibonaccis: Mystery and Magic in Numbers.* Palo Alto, Calif.: Dale Seymour, 1987.

Grossman, I. *Groups and Their Graphs.* Washington, D.C.: Mathematical Association of America, 1975.

Guy, R. *Unsolved Problems in Number Theory.* New York: Springer-Verlag, 1997.

Hallard, T. C., Falconer, K. J., and Guy, R. K. *Unsolved Problems in Geometry.* New York: Springer-Verlag, 1991.

Henle, J. *Numerous Numerals.* Reston, Va.: NCTM, 1975.

Hofstadter, D. *Godel, Escher, Bach: An Eternal Golden Braid.* New York: Basic Books, 1999.

Honsberger, R. *Ingenuity in Mathematics.* Washington, D.C.: Mathematical Association of America, 1978.

Honsberger, R. *Mathematical Diamonds.* Washington D.C.: Mathematical Association of America, 2003.

Honsberger, R. *Mathematical Gems II.* Washington, D.C.: Mathematical Association of America, 1976.

Hopkins, N., Wayne, J., and Hudson, J. *The Numbers You Need.* Detroit, Mich.: Gale Research, 1992.

House, P. *Mission Mathematics: 9–12.* Reston, Va.: NCTM, 1997.

Huntley, H. *The Divine Proportion: A Study in Mathematical Beauty.* New York: Dover, 1970.

Jacobs, H. R. *Mathematics: A Human Endeavor.* New York: W. H. Freeman, 1994.

Kline, M. *Mathematics in Western Culture.* New York: Oxford University Press, 1965.

Loomis, E. *The Pythagorean Proposition.* Reston, Va.: NCTM, 1968.

McKnight, C., et al. *Mathematics Education Research: A Guide for the Research Mathematician.* Providence, R.I.: American Mathematical Society, 2000.

Mottershead, L. *Metamorphosis—A Source Book of Mathematical Discovery.* Palo Alto, Calif.: Dale Seymour, 1977.

Nelsen, R. *Proofs Without Words.* Washington D.C.: Mathematical Association of America, 1993.

Niven, I. *Mathematics of Choice—How to Count Without Counting.* Washington, D.C.: Mathematical Association of America, 1965.

Niven, I. *Maxima and Minima Without Calculus.* Washington, D.C.: Mathematical Association of America, 1981.

Olds, C. *Continued Fractions.* Washington, D.C.: Mathematical Association of America, 1992.

Ore, O. *Graphs and Their Uses.* Washington, D.C.: Mathematical Association of America, 1992.

Ore, O. *Invitation to Number Theory.* Washington, D.C.: Mathematical Association of America, 1969.

Packel, E. *The Mathematics of Games and Gambling.* Washington, D.C.: Mathematical Association of America, 1996.

Petersen, I. *The Mathematical Tourist—New and Updated Snapshots of Modern Mathematics.* New York: W. H. Freeman, 2001.

Plichta, P. *God's Secret Formula: Deciphering the Riddle of the Universe and the Prime Number Code.* Boston: Element Books, 1997.

Posamentier, A., and Salkind, C. *Challenging Problems in Algebra 1.* New York: Dover, 1996.

Posamentier, A., and Salkind, C. *Challenging Problems in Algebra 2.* New York: Macmillan, 1970.

Posamentier, A., and Salkind, C. *Challenging Problems in Geometry 1.* New York: Dover, 1996.

Posamentier, A., and Salkind, C. *Challenging Problems in Geometry 2.* New York: Macmillan, 1970.

Ribenboim, P. *The New Book of Prime Number Records.* New York: Springer- Verlag, 1996.

Rouse Ball, W., and Coxeter, H. *Mathematical Recreations and Essays.* New York: Dover, 1987.

Rucker, R. *Mind Tools—The Five Levels of Mathematical Beauty.* Boston: Houghton Mifflin, 1987.

Schuh, F. *The Master Book of Mathematical Recreations.* New York: Dover, 1968.

Sinkov, A. *Elementary Cryptanalysis: A Mathematical Approach.* Washington, D.C.: Mathematical Association of America, 1980.

Sobel, M. *Readings for Enrichment in Secondary School Mathematics.* Reston, Va.: NCTM, 1988.

Steen, L. *Mathematics Today—Twelve Informal Essays.* New York: Springer-Verlag, 1984.

Stepelman, J. *Milestones in Geometry.* New York: Macmillan, 1970.

Stewart, I. *The Problems of Mathematics.* Oxford: Oxford University Press, 1992.

Stwertka, A. *Recent Revolutions in Mathematics.* New York: Franklin-Watts, 1987.

Tanton, J. *Solve This! Math Activities for Students and Clubs.* Washington D.C.: Mathematical Association of America, 2001.

Weinberg, S., and Goldberg, K. *Statistics for the Behavioral Sciences.* New York: Cambridge University Press, 1996.

Wills, III, H. *Leonardo's Dessert: No Pi.* Reston, Va.: NCTM, 1985.

Wisner, R. *A Panorama of Numbers.* Glenview, Ill.: Scott-Foresman, 1970.

Yates, R. *The Trisection Problem.* Reston, Va.: NCTM, 1971.

Zippin, L. *Uses of Infinity.* Washington, D.C.: Mathematical Association of America, 2000.

Internet Sources

The following Web sites may help you find a research topic or solve a problem. They will also be very helpful when you want ancillary material on a topic you have already chosen. Look for links to other potentially helpful Web sites, and use the "contact us" feature to write to the authors if you have a specific question or are looking for specific information. Remember that sites are continually being added and deleted. Ask questions and link aggressively—you never know where an excellent lead might be.

www.edc.org

http://forum.swarthmore.edu

http://mathforum.org/dr.math

www.nctm.org

http://camel.math.ca/Education/mpsf/

http://www.nrich.maths.org.uk/

www.mathpages.com

www.nationalmathtrail.org

Additionally, many sponsors of math contests publish books of their previous contests, with solutions. Some of these questions can be used as part of a paper to extend a topic or to find a topic.

Chapters 2 and 12 combine effectively to facilitate the selection of your topic. Once you have chosen a topic, you can use keywords from your topic to do Internet searches to find related material, different versions of proofs, and applications. These searches are often very productive and provide material that enhances the paper tremendously.

Chapter 13

A Guide for Instructors and Administrators

Math Research: Making the Case

If your school does not have a research program in place, you can use this book as a guide to writing a proposal for a new course, an academic club, or for a program ancillary to existing math courses. If you decide to have students write research papers, read the entire book before writing a proposal. Implementing a new course in a high school requires several levels of bureaucracy. A math research program is such a natural fit for so many of today's educational initiatives that creating a presenting on "Why does our school need math research?" is not a difficult task. Take note of some of these points you can include in your proposal for a math research program:

- Math research epitomizes the initiatives recommended by the Common Core Standards for Mathematical Practice, MP1 – MP8.
- The math research program naturally differentiates instruction since each student chooses their own topic and goes as far as their motivation, intelligence and effort will take them.
- Math research "raises the bar" for all students who participate.
- Success in math contests brings pride to the school. With so much of high school centered around success in sports, music and drama productions, it is nice to have an academic forum in which students so inclined can participate and get recognized for their excellence. Math contests and research presentations provide this forum.
- It is terrific that the mathematics education community has embraced the idea that all students should take algebra in high school. However, the intense efforts on this front must not lead us to overlook our bright, motivated mathematics leaders of tomorrow—we must allow them to mathematically stretch. As schools concentrate on ways to get struggling students to meet new state graduation requirements, the bright students are often not catered to—instruction is not differentiated to meet *their* needs. Most differentiation initiatives in schools center around students who find math difficult. This is because data on passing percentages is printed in all newspapers, and is closely watched by state education

departments. The talented, intrinsically motivated math students are our math leaders of tomorrow. They will lead this country's technological future. Their hunger for mathematical challenges cannot be ignored just because they are not an "issue" when it comes to a school's passing percentage on standardized tests. This educational malpractice sabotages the math-phile's entitlement to an education differentiated to meet his or her needs.

- Heterogeneously-grouped mathematics core classes can "rob" the talented math student of the daily high level of discussion and challenge they need to fuel their passion for mathematics.
- All students should be empowered with the commitment to and thrill of exploration and discovery in mathematics. One way to facilitate this is to assign a mathematics research paper.

How would your students describe mathematics and their mathematics education? The typical mathematics program covers, to some degree, units in algebra, geometry, trigonometry, probability, statistics, logic, graphing, precalculus, and calculus. This broad range of content may not give you the chance to explore any single topic in depth. If these units are not integrated, students will not get a feel for how different branches of mathematics can team up to help them solve a problem. Understandably, many people think that all of mathematics is "already discovered." The curriculum traditionally presents material in succinct, forty-minute packages that often do not reflect the way math evolved. The lessons are calculated, well prepared, and timed; the students know there will be an outcome by the end of the period. As a result, mathematical concepts may seem segregated, lifeless, and predetermined.

High school and college programs equip students with the tools (skills) necessary to do mathematics and some applications. Due to the content requirements and time constraints of the core curriculum, there may not be enough time devoted to allowing students the opportunity to discover, conjecture, and "play with" mathematics. Too often, "getting the answer" is the goal. This trend can suppress a student's natural curiosity. A math research program, based on the spirit of investigation and discovery that is the cornerstone of mathematics, can rekindle student enthusiasm. It also gives the math enthusiast an inside look at how mathematics is *really* done.

Math papers are sometimes assigned as a requirement in a core-curriculum math course, it is often completed in full outside of class without input from the instructor and other students. Sometimes it is an ancillary required project for students wanting honors credit for a course. Why are so many students asked to write math papers with absolutely no preparation? Why do some classes have a required math paper when no training toward completing this requirement is given? Why are instructors surprised when an assigned math paper becomes a math "book report" and a questionable learning experience? Is assigning a term paper tantamount to teaching research?

> *"The formal papers were another matter, however. I dreaded sitting down to read and grade the stacks of barely disguised, reconstructed encyclopedia essays. . . . The best solution, it seemed to me at the time, was to abandon the term paper."* (Countryman, 1992, p. 66).

Mathematics research papers address many of the recommendations of the Common Core State Standards for Mathematics Content and Practice, as well as NCTM and other professional organizations. Research skills are not innate; they are sophisticated, learned skills. They need to be taught, discussed, and practiced. Addressing math research formally is the first step in instilling good reading, writing, and research skills in your students. Avoiding the feelings of despair commonly experienced by instructors who assign papers requires a conscious attempt to teach about research. The purpose of this book is to introduce students and educators to a methodical way of learning how to read mathematics, write mathematics, and do math research.

When Do Students Write Math Research Papers?

Students equipped with the proper skills and motivation can write excellent mathematics research papers. Under what circumstances does a student write a math research paper?

- The student is in a standard, core mathematics course and does a paper as an enrichment project—perhaps for honors credit.
- The student is in an independent study program and, in accordance with school policy, chooses to write a math research paper for the major project.
- The student is enrolled in a mathematics research course elective in which the paper is the central project.
- The student is presenting a paper as part of a "graduation by exhibition" or International Baccalaureate program requirement.
- The student intends to enter a local or major mathematics contest.
- The student intends on majoring in math or science and wants to learn more about technical writing and/or oral presentations.

Preparation for Teaching Math Research

Most high school mathematics teachers do not have much experience with the independent study facet of a math research program. They are also not used to saying "I don't know" very often! This phrase is a frequent player in a math research course. Over the course of the year, the frustration of "not knowing" morphs into an infatuation with discovery, for both teacher and student. Teaching students the proactive, probing part of "what do you do when you don't know what to do?" is one of many great by-products of the program.

Starting a mathematics research program is an exciting undertaking. You will see your students reach new heights in their abilities to reason and communicate mathematically. A carefully designed program will breed enthusiasm for mathematics, pride, and quality workmanship. *Writing Math Research Papers* will help you begin your program. It is recommended that you read the Introduction and Chapters 1–12 carefully in order to immerse yourself in the student's frame of reference. Chapter 13 will help you with the nuts and bolts of setting up your program. There are many rubrics, suggested forms, and recommended procedures. Undoubtedly, you will make alterations and devise your own system along the way.

Acquire the necessary materials mentioned in Chapters 12 and 13. Choose the articles, problems, and other assignments you will be using initially so the first few weeks of the term are covered. It is helpful to have some articles selected, photocopied, and placed in a folder or binder that students can borrow when they are looking for a topic. The journals and articles suggested in Chapters 12 and 13 are a good source of topics. Ideally, each student should have access to a copy of these articles, which they can download or photocopy. Having the articles ready for the students facilitates their choosing an appropriate topic in an efficient manner. Use Chapter 13 as a checklist to guide your preparation. Try to make preparations based on the suggestions in Chapter 13 before the school year or semester begins. Above all, make sure that you and your students are primed to benefit from the research experience.

Research skills need to be taught formally as part of a research instructional program. Schools can create a schedule for research class meetings based on their budgeting and enrollment restrictions. A dedicated math research course is the best way to immerse students in the work of mathematics research. In this case there is ample time for instruction. Since research takes time and concentration, class meetings do not need to take place every day. One or two class sessions per week are usually enough. Some schools have programs that meet every other day. The research, editing, and revision processes require perseverance and time management, and are not conducive to daily homework. If students are writing papers as an ancillary part of a standard math course, the instruction could be a part of extra help sessions, or even an after-school academic club. Note that it is a little unrealistic to relegate the demands of a math research program to a part-time club or extra help arrangement.

A mathematics research paper is a major project that should be completed over a period of months, not days. Students involved in such a project should dive in with a "can do" attitude. This self-confidence will come from the students' perception that they are ready for the task at hand. Students must be equipped with the tools necessary for basic research. They are apt to be more interested in acquiring these tools if they see their paper as part of a "greater picture" that has purpose, is exciting and challenging, and is perceived as conquerable.

Experience with reading, writing, and problem solving is the necessary foundation. How can students gain this experience in preparation for writing mathematics research papers? They should read Chapters 1-12 and complete the activities included in these chapters. Here is a sample time line, which is very flexible and can change based on the number of class meetings the students have with their instructors.

Table 13.1. Sample Time Allotment

Month	Activities
1	Chapters 1, 2 and 12. Students pick a topic and articles for their paper. Students can do a search to find related material for their topic. Students start meetings with their teacher/mentors (consultations) about their specific topic.
2-3	Chapters 3-8, 11. Students engage in non-routine problem solving strategies. Students start keeping a journal/notebook of their research findings. Students start a Math Author Project, and learn about the formal parts of the paper. Consultations continue on research topic.
4-8	Chapter 9. The notes become formal drafts of the paper, and are revised frequently after consultations and critiques. A final paper is created. The oral presentation is started.
7-9	Chapter 10. Students prepare, finish, and give oral presentation. Students can choose a topic for their next year paper if they are doing one.

This is just one possible scenario. Different circumstances will require that you adapt. Being familiar with Chapters 1-12 will give you the global perspective you need to make your own adjustments to your program. Equipped with this information, students and teachers are fine-tuned for success and, with the instructor as coach, are empowered with the thrill and responsibility of exploration in mathematics.

The following has been adapted, with permission, from "Mathematics Journal Articles: Anchors for the Guided Development and Practice of Reasoning Skills" by Rob Gerver, published in NCTM's 1999 *Yearbook*. You can go to www.nctm.org to find out how to get a copy of the article.

Journal Articles—Providing a Focus

Well-written mathematics journal articles present students with a playing field on which they can develop and practice their reasoning skills. Typically, each article solves a specific problem and includes *some* information that is essential to the solution of that problem. Most of the best articles are only three to six pages long, but the topics they discuss can be built into yearlong projects for individual students or groups. The strength of the journal articles is that they offer the basic skeletal guidance students need to begin their own mathematical exploration and research. The student adds the *flesh*—extensions, explanations, examples, tests of claims, counterexamples, conjectures, and proofs—as the project grows.

The first page of a three-page article written by Amos Nannini for the November 1966 issue of the *Mathematics Teacher* is shown below. The article, entitled "Geometric Solution of a Quadratic Equation," is one example of the articles available to students that provide excellent exercises in reasoning. The article explains a construction that can be used to find the roots of a quadratic equation.

GEOMETRIC SOLUTION OF A QUADRATIC EQUATION

By AMOS NANNINI

Milan, Italy

MAY I remind the reader of an elegant construction, discovered by Descartes, whereby it is possible to find both solutions of a quadratic equation $ax^2 + bx + c = 0$, geometrically? [1][1]

Given a rectangular coordinate system, mark off on the y-axis a segment OA of length 1; on the x-axis a segment OB of length $\frac{-b}{a}$ (of course in the same unit chosen for the y-axis), and, on the perpendicular to the x-axis through B, lay off a segment BC of length $\frac{c}{a}$. (Fig. 1.)

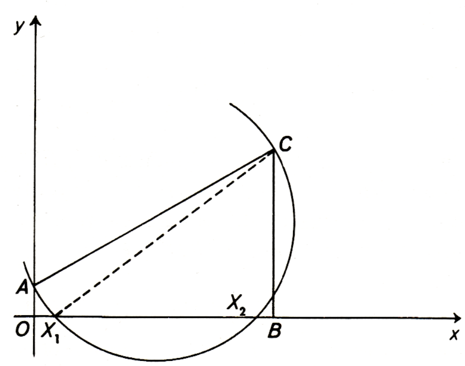


FIGURE 1

We assert that, if the circle having AC as its diameter intersects the x-axis in the points X_1, X_2, then the lengths of OX_1, OX_2 are the solutions of equation

$$ax^2 + bx + c = 0.$$

Indeed, since angle AX_1C is right (being inscribed in a semicircle), right triangles OAX_1, X_1BC are similar, because they have angle AOX_1 = angle X_1BC (right angles), and angle OAX_1 = angle CX_1B (both complements of angle AX_1O); remember that, since angle $AX_1C = 90°$, it is necessary that angle AX_1O + angle $CX_1B = 90°$.

Hence the proportion between corresponding sides,

$$OA:OX_1 = X_1B:BC,$$

or, keeping in mind their measures,

$$1:OX_1 = \left(\frac{-b}{a} - OX_1\right):\frac{c}{a},$$

whence, equating the product of extremes to the product of means and transposing all terms to the left,

$$(OX_1)^2 + \frac{b}{a}OX_1 + \frac{c}{a} = 0,$$

and, multiplying through by $a(\neq 0)$,

$$a(OX_1)^2 + b\cdot OX_1 + c = 0,$$

which proves our statement.

An analogous argument holds for OX_2, and also for coincident X_1 and X_2. In this latter case, the circle would be *tangent* to the x-axis, and the equation would have a double solution $OX_1 = OX_2 = \frac{-b}{(2a)}$. (Fig. 2.)

[1] Numerals in brackets indicate references given at the end of the article.

Before reading the article, students should review the ways they already know to solve a quadratic equation. These may include factoring, using the quadratic formula, and finding the *x-intercepts* of the graph employing a graphing calculator. If these methods are unknown to them, students can learn how to complete the square by following the steps in an algebra textbook. At this point, they can study the derivation of the quadratic formula, which involves completing the square. They can be given the steps for completing the square and asked for the reasons for each step or be given nothing but a few hints about the process.

Extending "Between the Lines" in Mathematics: Extensions Within

Read the first page of the article. Notice that instructions are given without any reasons. Determining the why behind each one of the steps is a terrific exercise in reasoning. Students might need hints, to different degrees, for some steps. All this reasoning is "between the lines." A comprehensive reading of the article will require much student input and teacher feedback. Since the target audience of the article has a mathematics background and any article may have space limitations, not everything in the article is explained. In Nannini's article, take note of some of the explanatory material that needs to be inserted to produce a paper that has a fully explained, logical flow:

1. The article uses the problem-solving strategy "work backward" to build the construction. The construction is used to *find* the roots, yet the roots themselves are plotted early in the construction, as explained in the second paragraph.

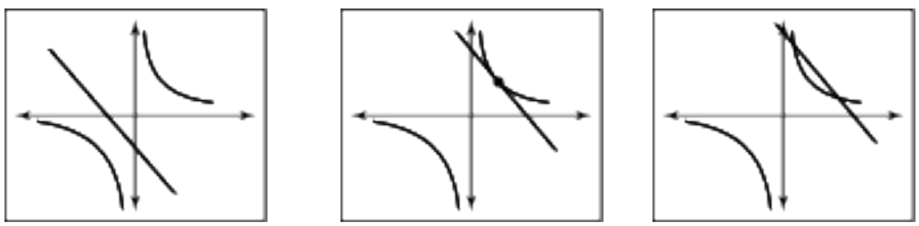
2. The two *x-intercepts* do not determine a unique circle, therefore three noncollinear points must be used. How do three noncollinear points determine a unique circle? This will require students to tap into what they know about chords and perpendicular bisectors.

3. Where do the expressions $-b/a$ and c/a come from? Teachers can direct students to pick several quadratic equations—first some with leading coefficient 1; then others without—and find the roots. If a spreadsheet is available, then students can look for the patterns suggested by the sum and product of the roots, using the data in the spreadsheet. After making conjectures about the sum and product of the roots, students can prove their conjectures using the form of the roots as expressed in the quadratic formula.

4. The coordinates of ***B*** and ***C*** can be determined if the axes are viewed as secants to the circle. *If two secants are drawn to a circle from a common point, the product of one secant and its external segment is equal to the product of the other secant and its external segment*. Students unfamiliar with the theorem and students who need to revisit the theorem can read about it in a geometry textbook.

5. The other point where the circle intersects the *y*-axis can be called point D. The coordinates of D, (0, c/a), can be found using the properties of secants. The image of D over a vertical line reflection that bisects the circle is point C. Therefore, C's y-coordinate, c/a, can now be determined. Students will need to recall information on inscribed angles to see why AC is a diameter, as claimed in the bottom of column 1 of the article. The coordinates of point B can be found using the reflection of the origin in the same line and the sum of the roots.

6. The construction claims to find the roots by using the sum and product of the roots. Is it possible that two other numbers, besides the roots, can yield the same sum and product as the roots do? If s and p represent the sum and the product, respectively, an examination of the graphs of $s = x + y$ and $p = xy$ is necessary. Using a graphing calculator, students can see that one gives a negatively sloped line and the other a rectangular hyperbola. There are 0, 1, or 2 intersection points, as shown. The two roots are the only two numbers that will yield the given sum and product.



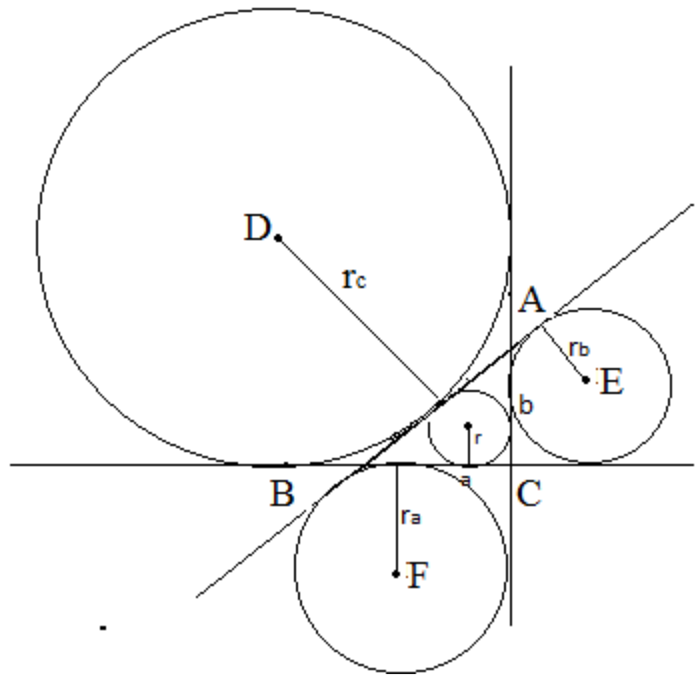
7. Students need to use their algebra and plane geometry tools to follow a logical argument as they read the material in the second column.

Notice how much reasoning was required to do a comprehensive interpretation of every facet of just the first page of this article. Observe how technology (spreadsheets and graphing calculator) enhanced the treatment of the article. Note that the article was written before calculators and computers were in schools. The article presents the one-root case, the negative/positive-root case, and the imaginary-root case. Students must create new, analogous, but different diagrams and proofs for these cases. These proofs are guided—they are not 100 percent original—since they are rooted in the first proof in the article. However, there are differences. Students must be able to understand and use the building blocks of previous proofs before attempting original proofs from scratch. Many articles can be attacked in a similar fashion.

Extending the Article with Original Research: Extensions Beyond

Students who work quickly or want to continue their research beyond the initial article (even in subsequent courses) can read two articles that have some common thread and attempt to combine the results. With the help of all of the guided reasoning exercises in which they are engaged while reading their first article, the second article and the combination of two articles can be handled with much more independence. Many students have taken this route in the past. Here is an example of how Alex extended her initial article.

Alex read the article "On the Radii of Inscribed and Escribed Circles of Right Triangles" from the September 1979 *Mathematics Teacher*. Escribed circles are tangent to one side of a triangle and the extensions of the other two sides. The figure below shows a right triangle with its inscribed circle and its three escribed circles.



The article derives expressions for the radii of the inscribed and one escribed circle to a right triangle. Alex extended the results to include circumscribed circles and the two other escribed circles. This was a natural extension since the article gave formulas for the radii of the other two escribed circles but did not show how these formulas were derived.

After completing the article, Alex found a second article, "Pythagoras Meets Fibonacci," from the April 1989 *Mathematics Teacher*. This article shows how Pythagorean triples can be generated from four consecutive Fibonacci numbers. Alex intuitively felt that she could combine both articles to find a relationship between the original Fibonacci numbers used to create a right- triangle triple and the resulting radii of the escribed circles to the same right triangle. First she decided to see if she could notice a relationship between the Fibonacci numbers and the radii by trying about a dozen numerical examples and looking for a pattern. She did her work on a spreadsheet and made conjectures about the formulas based on several patterns she found. She proved her conjectures using algebraic substitution and found that they were, indeed, correct.

This line of reasoning is typical of many undertakings in the mathematics research arena—students find a pattern by creating and analyzing numerical examples; they make a conjecture about a general relationship based on the pattern; and they set out to prove or disprove their conjecture. Notice that the articles first provide a foundation and then act as a springboard for research and exploration. Creating a bridge between two articles is a common way to find extensions beyond the original articles.

The extensions within and the extensions beyond provide the gist of the course. The depth of the extensions will depend on the student's motivation, effort, and math ability. You can take many steps to optimize student output, and some of these will be covered in the next section.

Planning Course Logistics

As you embark on your research program, keep in mind that you will have some bumps along the road as you strive to meet your long-range goal of a solid research program. As you do your yearly budget planning, add a little to the program, and in several years you will have an impressive, growing program. Look for local, state, and national grants that can be used to secure money to pay for your program's needs. Here are some logistical suggestions:

Get each student's e-mail address. Set up an address book on your computer. This is a great system for sending notes and reminders to students you don't see every day.

Teach equation editor use and writing tips with an interactive whiteboard, or computer and monitor or LCD screen. Your monitor or screen will need to be large enough for the entire class to see. You can take students into a computer lab if you cannot get a monitor or LCD screen in your classroom.

Provide a mailbox for each student and yourself. Use the mailboxes to "trade" drafts and revisions, and keep copies of readings and articles. High school instructors can use mailboxes in their classroom; college instructors may be able to set up a mailbox system in their own office or in the math department office. Inexpensive corrugated cardboard mailboxes are available in office supply stores and educational catalogs.

Provide a schedule for consultation time slots. The ideal forum for the math research undertaking is to set up a dedicated math research course. Many different scheduling arrangements are possible. Classes can meet as a whole in many different arrangements, depending on school scheduling restrictions and priorities. One requirement of any program is that students have regular consultations with a teacher/mentor to discuss progress on their individual topics. The logistics of this arrangement must be carefully orchestrated. You may need to customize some of the Chapter 13 suggestions so they work within the restrictions of your academic schedule. Just remember that good organization is essential for teacher and student!

The independent study nature of a math research program requires that every keep careful track of consultation times and assignments. The individualization means that a students cannot "call up a friend to get the homework" if they are missing something. So students need to keep careful notes on what they are required to do each week, and teachers and students need to keep careful track of consultation time slots.

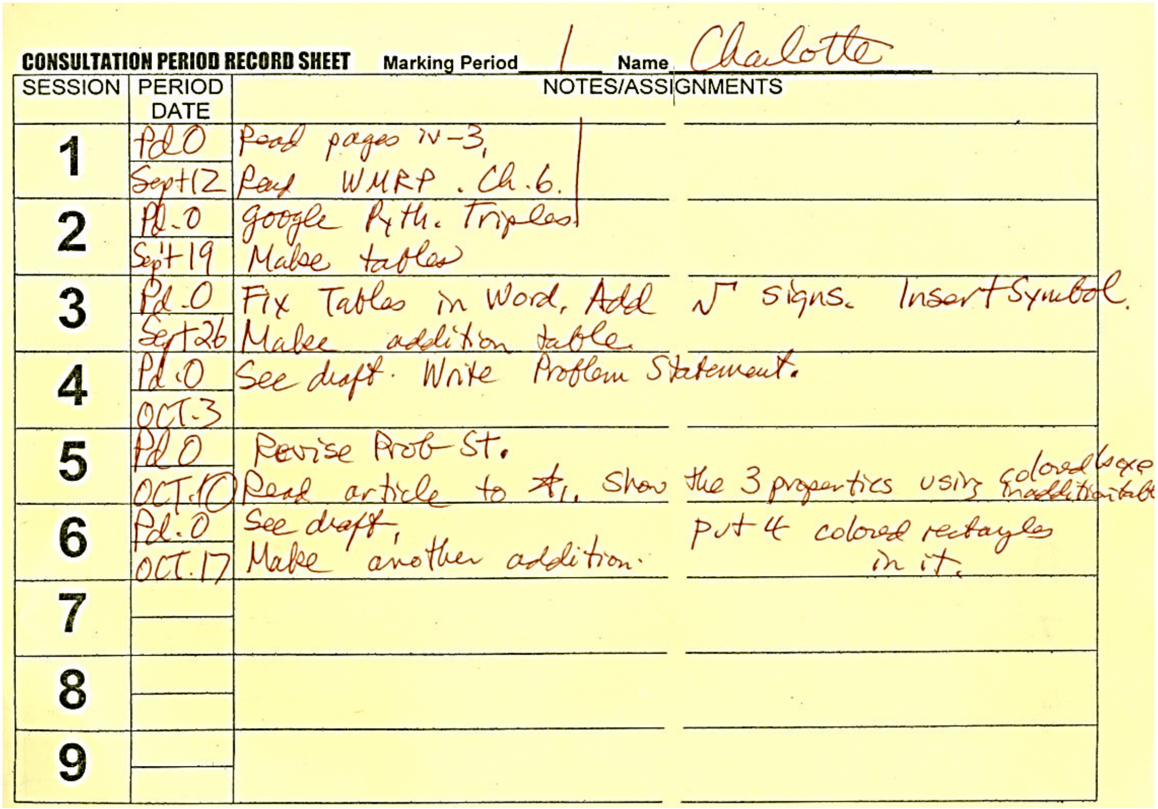
To facilitate this, keep a centrally located calendar easily available so students can sign up to see you on your mutual free periods. You can download a calendar from the Internet or just make your own with a word processing program. For a small number of students, you can fit the calendar on a regular sheet of paper. The following calendar was enlarged at an office supply store, since it is for a program that has 40 students, and more room was needed.

The following shows an enlargement of one day of the calendar. Notice this teacher/mentor has four time slots available in the morning. The teacher is free periods 3, 5, 7, 9. Each free period has been divided into three time slots, A, B, and C. The length of a time slot can vary depending on period length and the number of slots. Consultations should minimally be ten minutes. Be sure to block out any time slots you need for your own personal prep time, lunch, and other activities. The advantage of having the time slots for each day, as opposed to just a blank box for each day, is that two students cannot accidentally sign up for the same time slot.

DAY OF MONTH	DAY OF CYCLE
7:00	5B
7:15	5C
7:30	7A
7:45	7B
3A	7C
3B	9A
2B	9B
3C	9C
5A	AS

Provide students with a consultation record sheet. At each consultation, teachers give weekly assignments. Students keep track of the assignment on a Consultation Record Sheet, as shown below. This allows teachers and students to recall what was assigned at the student's previous meeting. Students use this as a "homework" assignment pad, and teacher/mentors use it to recall what the student was supposed to do for the subsequent consultation. The sheet is easy to create on a computer. You may want to photocopy them onto card stock since they need to last an entire marking period. Double side them so the following marking period can use the same sheet. After the second marking period, new sheets can be given out for the remaining marking periods.

Consultations are generally held weekly. Therefore, there are 9-10 consultations in a marking period of four quarters. Students can always opt to come extra times, if the teacher has open slots. Student work is completed on a weekly basis. Be sure to assign a reasonable amount of work for the week. Some students might have trouble just completing the work, while others will complete it and extend to other things. The ball is in the student's court. Be sure to take note of students who are habitually underprepared. See the consultation assessment criteria in the Assessment section of Chapter 13 to see how consultations are evaluated. Show them the consultation grading criteria and ask them to evaluate themselves so they see what is required. This exercise often makes a difference.



CONSULTATION PERIOD RECORD SHEET Marking Period 1 Name Charlotte

SESSION	PERIOD DATE	NOTES/ASSIGNMENTS
1	Pd 0 Sept 12	Read pages iv–3, Read WMRP. Ch. 6.
2	Pd. 0 Sept 19	Google Pyth. Triples Make tables
3	Pd. 0 Sept 26	Fix Tables in Word, Add √ signs. Insert Symbol. Make addition table.
4	Pd. 0 OCT. 3	See draft. Write Problem Statement.
5	Pd 0 OCT. 10	Revise Prob St. Read article to #1. Show the 3 properties using colored boxes in addition table
6	Pd. 0 OCT. 17	See draft, Make another addition. put 4 colored rectangles in it.
7		
8		
9		

Start a research library. Subscribe to the journals listed in Chapter 12. You can keep journals in corrugated cardboard magazine filers, organized by date.

Having journal articles easily accessible to students is a real advantage. Instructors should begin a bank of articles by keeping a copy of all student articles in a binder or other organizational device. As students find good Web sites, these should also be included on a list in the binder. This cooperative venture between students and instructors benefits all students who write research papers. Eventually, dozens of articles will be readily available in this binder for use by many students. You may even decide to network with other instructors or school districts to pool your materials. Use Web site bulletin boards and other public forums to locate collections of old math journals. Retiring mathematics instructors often have extensive collections they are willing to donate. Keep one copy of each textbook you own in this library, too. Students can use these when they need to learn a new skill for their papers.

Display student posters and manipulatives. If you are a high school instructor, you can hang posters on classroom walls so students can see what is expected of them. Hanging up awards and certificates students have won will make the room look great!

Over the years, as your research program becomes better equipped, it will be a focal point for the students in the program. They will use the resources and the mailboxes extensively.

If you are a college instructor, bring several posters and other visual aids to class at an appropriate juncture in the course when presentations and visual aids are being discussed. If space allows, you might consider having these visual aids available in your office for students to access. You might even have a former research student present his or her project to the class as an example of what you will be expecting from your current students.

Make appropriate budgetary arrangements. Formal programs require financial support because they count as class assignments in an instructor's schedule. All programs need books, audiovisual materials, contest entry fees, poster materials, access to technology, mentor time, and so on. If your school decides to offer a formal math research program as part of instructors' schedules, you'll need some time to develop the program and gather the program materials. If financial support becomes an issue, you should consider options for raising funds. Become aware of local, state-level, and national grants you may be eligible to apply for. Establishing a research course could make you eligible for a grant. Your local or state mathematics teacher organization may also offer funding. Read their newsletters and journals regularly to find announcements of such opportunities. Read the Making the Case section of Chapter 13 carefully because Board of Education and administrative support and approval depends on a solid, convincing appeal for the new course.

Once your course is approved, you'll need to find teachers excited about teaching it. Usually, these teachers would have been the core of the "making the case" stage of the proposal. The program needs a teacher who is optimistic and excited about the challenge!

The Instructor as Mentor/Coach

Many students have become aware of the fact that they are approaching mathematics in all their courses differently than previous generations approached it. The instructor's role is also changing; perhaps your math department has taken several initiatives in the areas of Common Core, technology, differentiated instruction, alternative assessment, problem solving, portfolios, and so on. In this section, we specifically address the instructor's role as coach in math research. As a coach, you want steady progress from your students. To help ensure that progress, you will have periodic consultation sessions with your students, edit each draft of their research papers, and assess the major areas of their project.

Consultations

Students research and write their papers primarily on their own time. (There may be a few instances in which students work on their papers in a classroom situation.) The instructor acts as coach. You must make sure that your students make consistent, gradual progress on a timely basis and that they improve their reading, writing, and research skills. Students need guidance and deadlines to accomplish these goals. For this reason, an essential part of any research program is private consultations between instructor and student. Individual students should meet periodically with a mentor (usually the instructor) and discuss the progress on their article. A consultation time slot is an extra—but mandatory—help period devoted to the paper. (Students might also come for extra help on the paper during your designated free period or office hours.)

The consultation session is scheduled jointly by the student and instructor to take place during a mutually agreeable time slot. The length and frequency of consultations will depend on both student need and your schedule—but make sure you conduct them as regularly as possible. Some students can touch base with you for five minutes every two weeks to discuss material. Other students may need more time. Consultations should be scheduled so as to ensure that each student is progressing at a reasonable pace. You and your students must commit to this key feature of the research program. The consultation procedure requires the student to engage in research between sessions, preparing material for each consultation. Newly completed work must be brought to each consultation; otherwise, the instructor has the right to schedule a new meeting.

The student discusses his or her new work at each consultation. As a result, over the course of the entire project, the student has worked extensively on the research in many separate sittings. This process is the antithesis of the infamous "paper started the weekend before it is due" syndrome. The strength of the consultation period is the one-on-one communication—there is an onus on the student because there is "nowhere to hide," unlike a normal classroom setting. This raises students' level of responsibility, their expectations of themselves, and, as a result, their achievement. The consultation time is not an extra, a frill, or a dispensable luxury—it is an essential part of a successful research program. Holding consultations weekly is ideal, but the frequency depends on teacher/mentor availability.

What happens at each consultation? What is your role? You assign readings from the article(s) the student picked. Make sure you assign appropriate amounts of reading; a paragraph in mathematics can be very involved. Students come to the next consultation with the material read, underlined, and annotated directly on their copy of the article. They should have taken notes that ask questions, explain material from their article, make a conjecture, test a claim, try a proof, or extend the reading passage. To keep track of their current assignment, each student keeps a Consultation Record Sheet, a shorthand journal of their tasks for the week.

The Consultation Record Sheet can be filled in by the student and/or instructor during the consultation period. It clearly delineates what needs to be done before the next consultation. Some items entered on the sheet will be for future reference and not for the very next session. The record sheet setsup a reasonable time line for the students, which is a helpful tool studentscan adopt and adapt for later use, and you will have a way to keep them progressing in a logical manner with a good deal of accountability. Consultation Record Sheets from previous marking periods should be saved by the students; they usually contain suggestions for the future that students will need.

Have your own copy of each student's article available for the consultation. At the beginning of each consultation, reacquaint yourself with the student's progress using the consultation sheet. Go over the student's work, ask questions, answer questions, and so on. Discuss the material and assign the next reading. Force students to be explicit in their explanations; this will improve their oral and written communication skills. Students can record the discussions on their cell phones. The consultation allows you and the student to have a mathematical discussion that will raise the level of the student's content knowledge and communication skills. It is terrific preparation for an oral presentation as well as an excellent forum for witnessing the combination of student empowerment and accountability.

Editing Drafts of the Research Paper

When the students begin formal writing of their individual papers, they will be writing up material they are familiar with. They will devise an outline and follow the writing recommendations in Chapters 4- 6 and Chapter 9. They should get in the habit of saving their paper file in at least two places. You can have students e-mail drafts to you and save them on the school computer just in case a student's computer crashes. As they produce pages, they will submit them to you for editing. Papers can be submitted at consultations or at any other agreed-upon time. Drafts should be submitted a few pages at a time to allow you to read and return them in a realistic length of time. Advise students to have disk and hard-copy backup if they are leaving drafts in your mailbox. As you read the papers, give full-sentence suggestions, make corrections, and ask questions. Acquaint yourself and your students with the editing symbols shown in Chapter 3. These shorthand comments take care of the more mundane corrections. Other comments will require phrases or full sentences and should be made either in the margins or between the double-spaced lines of text. Be sure you check the mathematics as well as the writing.

You can return edited drafts in class, at the consultations, or by placing the edited drafts in students' mailboxes. Students will revise their papers according to your editing, and hand in the revised material along with any new material. This constant trading continues until the paper is finished. You can discuss the edits at the consultations. Editing suggestions should appear on the draft, not on the Consultation Record Sheet. Students must save all drafts. Sometimes old drafts are needed, especially if material is lost on the computer. When the paper is completed, it will not be a "new" reading for you; you will know the material and the writing well because you will have seen it so often. As the coach, you played an important role in the growth of the paper, and you will be assigning a grade for the work done. How will you evaluate the different facets of the research process? Guidelines are given in the next section.

Assessment

Grades are based on problem-solving exercises, homework, paper writing, class participation, and consultation quality. Formal tests and/or take-home tests can be given at the instructor's discretion, and they can include questions about problem-solving theory and/or research techniques. The time limitations of in-class tests hinder your ability to give non routine problems. You may give problem-solving tests to students in cooperative groups, even over a period of days. Such tests could involve a multistep application problem that serves as a performance assessment.

Grading short-answer questions is generally considered uncomplicated. Grading geometry proofs and long questions where partial credit is given for work shown requires more thought. Assessing problem solving and research requires a balance between holistic evaluations and grading specific content. The large amount of quality contact time the research instructor has with each student optimizes the instructor's ability to make a reasonable judgment of the student's progress. We will discuss assessment in five areas:

1. Problem-solving skills
2. The Math Author Project
3. Consultations
4. Written papers
5. Oral presentations
6. Reflective assessment

Your research program may not include traditional tests as indicators of progress, and rightfully so. Problem-solving and math research skills are usually not timed; their quality is measured by the final product, not by how the work appears after a forty-minute period. Students need to know how they are doing, so make it clear what will be graded. You can base students' grades ona combination of the five areas listed above, but note that all five may not be part of every marking period. The relative weights of each area are determined by you; make sure they reflect the degree to which the area was stressed during each marking period. Students should be made aware of the specific grading criteria that will be used. We give some suggestions for grading in the five areas; feel free to adapt them to realistically judge the priorities of your particular course.

Assessing Problem-Solving Skills

Grading non-routine problems given in homework, in class, or on tests takes more time than grading short-answer or traditional "show all work" problems. Because the grader can't be a mind reader, students must be required to explain each step they take in solving such problems. Grading scratch work that lacks verbal explanation requires the grader to infer what the student was thinking, and this is not a desirable situation in making evaluations. Several books on teaching problem solving include a scoring scheme that makes grading problems more objective. Some are included in Chapter 13. If a book is out of print, you can do an Internet search and try to find a used copy from a used book website.

The communication skills learned in Chapters 4 – 6 play an important role here. It is often difficult to figure out what a student was thinking if you see some scribbled numbers and/or diagrams on a piece of paper. Explaining how a problem was solved gives the student more practice in technical writing, and this should be encouraged. You can have a student solve a problem, explain it fully in writing, and then have another student who did not get the problem explain the solution to the class.

Ted Herr and Ken Johnson (2001) adapted a grading strategy suggested by Randall Charles in *Problem Solving Experiences in Mathematics* (1986). We use Herr and Johnson's adaptations to create a list that consists of five 2-point criteria. Each problem is then worth 10 points, and students can simply multiply by 10 to get a percent to judge their problem-solving ability. The numbers indicate the number of points that should be awarded.

PROBLEM SOLVING ASSESSMENT: 0, 1, or 2 IN EACH CATEGORY

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1. ___Does the student understand the problem?

0: Student misinterprets the problem completely.
1: Student misinterprets part of the problem.
2: Student understands the problem completely.

2. ____Does the student s choose a reasonable problem-solving plan?

0: Doesn't use any discernable strategy.
1: Uses an inappropriate strategy.
2: Chooses a correct strategy that would suffice to solve the problem.

3. ____Does the student correctly execute their plan?

0: Doesn't execute any plan, or uses the chosen strategy incorrectly.
1: Uses the chosen plan, but with some errors.
2: Uses the plan without any errors.

4. ____ Does the student answer the question?

0: Doesn't give an answer.
1: Gives an incorrect answer due to an error in previous steps, or gives a correct answer based on erroneous information.
2: Gives a correct answer, based on correct information, in a full sentence.

5. ____Explanation

0: Offers no explanation in full sentences, or is too vague.
1: Gives explanations of all the steps taken with some errors, or is incomplete.
2: Gives a comprehensive, precise explanation.

For each problem, the grade consists of a total followed by an ordered set of five numbers that correspond respectively to 1, 2, 3, 4, and 5 above. You might want to look at other suggestions for grading problem solving and/or adapt this outline. Show your grading criteria to your students to helpthem understand their grades better and concentrate on areas that need improvement.

Assessing Math Author Projects

If you teach a separate, dedicated Math Research course, you can align your Math Author Projects with the work the students are doing in their "core" mathematics classes. The Math Author Project, featured in Chapters 4 - 6, allows students to begin to practice and receive comments on their writing. This practice will make the paper writing a smoother process. Students earn grades from 1 (lowest) to 10 (highest) for each criteria:

MATH AUTHOR PROJECT ASSESSMENT: 1-10 IN EACH CATEGORY

1._____The paper covers all of the notes.

2. _____The paper is mathematically correct.

3. _____The paper is well organized with respect to sections and paragraphs.

4. _____The physical layout of the paper, including diagrams and tables, is high quality.

5. _____Intricate diagrams developed in class are graduated where necessary.

6. _____The material is explained well (proper sentence structure and correct math terms).

7. _____Related homework examples not from the class notes are included.

8. _____The steps of proofs and derivations are adequately explained.

9. _____The examples done in class are annotated with purpose.

10. ____Good explanations of typical pitfalls are given.

11. ____Questions from handouts, tests, and quizzes are included and analyzed.

12. ____A list of key terms, correlated with page numbers in the textbook, is given.

13. ____ Calculator keystroke sequences and computer programs are explained.

14. ____Technology is used with discretion.

15. ____The list of writing tips from Chapters 4 - 6 was followed.

16. ____The project was submitted, edited, and revised in a timely fashion.

17. ____All recommendations and corrections from edits are incorporated into the paper.

18. ____The project does a clearer job of explanation than the original notes.

19. ____The general depth and quality are commensurate with the student's ability.

You can convert the raw score into a percent and enter a grade in your grade book for the Math Author Project. Add or delete items to make this list accurately reflect your priorities. Students may complete more than one Math Author Project if time permits. They are great "extra credit" assignments since they help students learn required course material.

Assessing Consultations

Students should be made aware that their performance at consultations will be assessed at the end of the term. Individual consultation sessions are not graded. The grade is a holistic grade that reflects effort, depth and quality of questions, depth and quality of extensions, evidence of original thought, work with proofs, article annotations, testing of claims, quality of notes, and punctuality. Students earn grades from 1 (lowest) to 10 (highest) for each of the following criteria:

CONSULTATION ASSESSMENT: 1-10 IN EACH CATEGORY

---

1. ____Weekly consultation record sheet goals are met.

2. ____There is evidence of time and effort spent since the previous consultation.

3. ____The last draft (of paper or transparency masters or PPT) was revised.

4. ____There is evidence of probing and persistence.

5. _____Articles and previous drafts are annotated with corrections.

6. _____There are original extensions of the article.

7. _____There are extensions of the recommended revisions.

8. _____Notes were taken.

9. _____Material was reworded.

10. ____Oral presentation materials were prepared.

11.____Student attended required number of consultations for the quarter.

12.____Student had all materials necessary for each consultation.

13. ____Miscellaneous. Explanation: ________________________

You can convert the raw score into a percent and enter a grade in your grade book for consultations. Add or delete items to make this list accurately reflect your grading criteria. Naturally, you will uncover differences in the level of the students' work. Some students will exceed expectations, while others will fall short in some areas mentioned here. Remind the students that their grade will be based not on occasional exams but on performance as monitored consistently throughout the marking period. In this respect, it is a very realistic indicator.

Assessing Research Papers

For the research paper grade, students earn grades from 1 (lowest) to 10 (highest) based on your evaluation with respect to each of the following statements:

RESEARCH PAPER ASSESSMENT: 1-10 IN EACH CATEGORY

---

1. _____ Drafts were updated consistently and regularly, a few pages at a time.

2. _____The abstract is clear, succinct, and comprehensive.

3. _____The problem statement is well-developed from the familiar to the new material.

4. _____The mathematics is correct.

5. _____The paper does a clearer job of explaining the topic than the original article does.

6. _____There are original conjectures made.

7. _____Proofs in the article are explained thoroughly.

8. _____There are original proofs.

9. _____There are original extensions of some of the ideas in the article.

10. ____Diagrams and/or tables are graduated where necessary.

11. ____Color is used appropriately.

12. ____Mathematical notation and terminology are used correctly.

13. ____Captions for figures are descriptive and formatted correctly.

14. ____Table headings are descriptive and formatted correctly.

15. ____Uploaded diagrams are given a citation in the bibliography and in the caption.

16. ____The addendum reflects an appropriate amount of work for the allotted time.

17. ____The physical layout of the paper--text, diagrams, tables—is high quality.

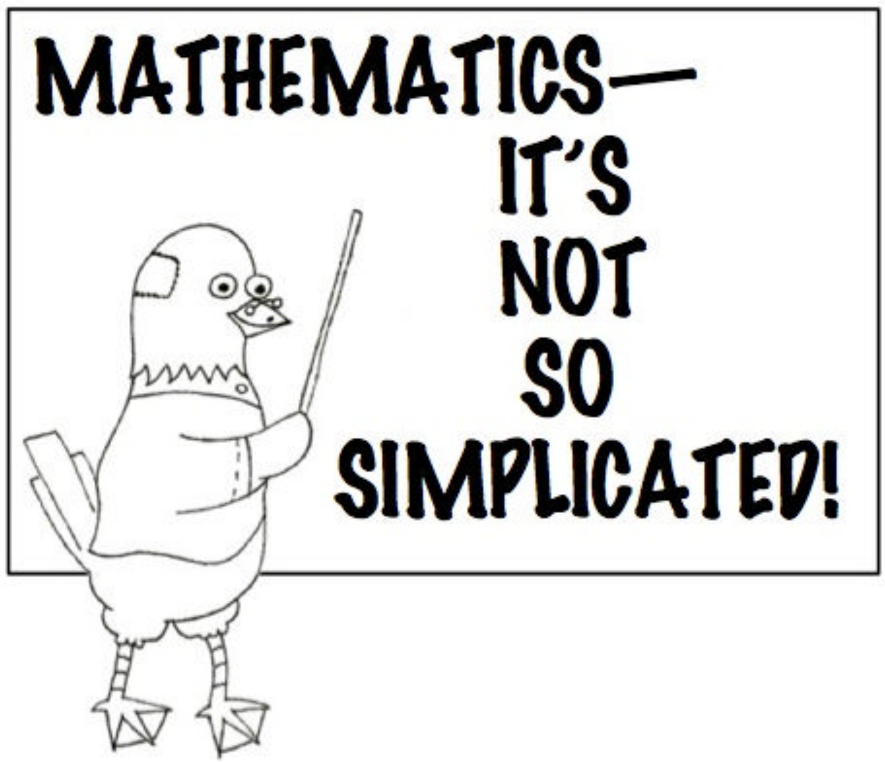
18. ____Appropriate and sufficient examples are given.

19. ____Recommendations for further research flow naturally and are well-thought out.

20. ____All edits and recommended changes are incorporated into the paper.

21. ____The depth and quality of the paper are commensurate with the student's ability.

You will be very attuned to each paper when the final version is handed in because you will have edited it extensively and discussed it in the consultations. Your last reading of each paper is perhaps the easiest, since all the comments you made previously have been incorporated. Keep in mind that you are grading a final paper—try to judge the work itself. It may have required more effort and more pain for some students to get to the same level as other students. If the paper's topic was appropriate for the student, then there is no penalty or reward for doing many drafts and revisions. The consultation grade can be adjusted to reflect that effort and commitment. The extensive editing over a period of months should "weed out" any problems, and the resulting paper should be high quality. You might want to weight the last question to count for more than 10 points, for example, from1 (lowest) to 20 (highest). Compute the raw score, convert it to a percent, and enter a grade.



Assessing Oral Presentations

When the papers are completed, you can have your students give oral presentations, as discussed in Chapter 9. The oral presentations can be graded independently of the paper and the consultations. For the oral presentation grade, students earn grades from 1 (lowest) to 10 (highest) based on your evaluation with respect to each criterion. The criteria are divided into three categories—preparation, delivery, and effort. Feel free to adjust them any way you'd like.

ORAL PRESENTATION ASSESSMENT: 1-10 IN EACH CATEGORY

PREPARATION

1._____ Posters and slides have large enough print and bold enough lines.

2._____Posters and slides use color where appropriate, to help convey mathematical thoughts.

3._____Posters and slides have titles.

4._____Difficult-to-remember last lines of slides are repeated on subsequent slides.

5._____There are divider slides.

6._____ Transparencies are in a binder with the paper Power Point slides printed as handouts.

7._____The teaser is aesthetically engaging, and is doable in under 60 seconds without a calculator.

9._____Crucial tables and diagrams are always in view when needed.

10._____Powerpoint slides have animation where appropriate.

DELIVERY

11.____The introduction is appropriate for the novice. Problem is clearly stated.

12.____There is no dependence on the reading of slides to recall information.

13.____Inflection is used as "verbal italics."

14._____Voice projection is used as "verbal italics."

15._____Vocal work transcends "tell" to "sell."

16._____The pointer is used in a "touch and still" format.

17.____The pace is appropriate.

18.____ Eye contact is frequently made.

19. ____There is well-organized board work and/or "live" written work on slides or posters.

20.____Physical transitions from topic to topic are smooth. Appropriate segues are used.

21._____ Pausing is used frequently, appropriately, and effectively. Time is allocated wisely.

EFFORT

22._____Every single slide and poster was seen and critiqued before the Math Fair.

23.____Manipulatives were used where appropriate or suggested.

24.____ Preparation of all materials was done timely, in advance.

25.____The oral preparation and presentation had punctual, appropriate effort.

26.____There is evidence of rehearsal time spent at home.

27.____Preparation included a series of rehearsal/critiques/revisions done sufficiently in advance.

Above all, try not to let the student's performance on the paper and the consultations predetermine the oral presentation grade. You may find that a student with an excellent paper needs work on the oral presentation. Other students may surprise you with excellent visuals, good organization, and a smooth delivery. A copy of a blank grading sheet can be given to students so they will be aware of the grading criteria. Students in schools that require students to graduate by "exhibition" may be able to use their oral presentation of their math research to satisfy this requirement.

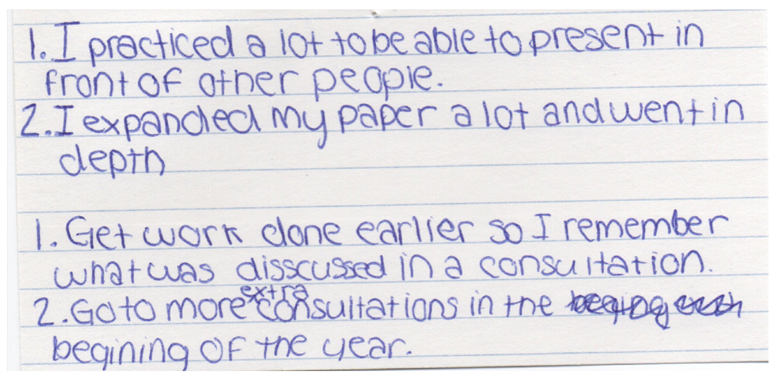
Reflective Assessments

At the conclusion of the year, students have really completed an admirable amount of challenging work, so it's no surprise that a sense of pride is evident as the school year ends. At this point, it is a good idea to have students do the only written reflective piece they will do all year.

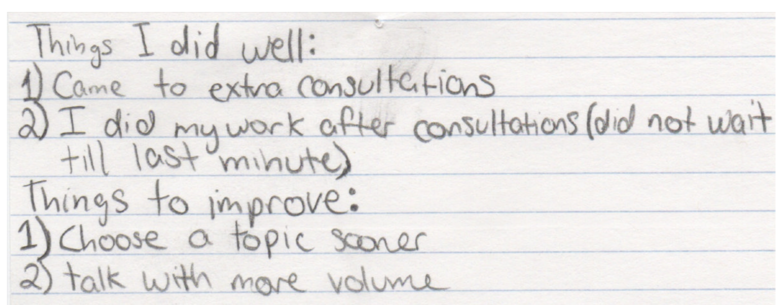
Each student should submit a short list divided into two categories:

- Tasks I did well this year
- Tasks I should improve upon

Often, writing assignments in a "regular" math class are purely reflective. The research paper itself does not have a reflective component—its mission was to immerse students in research, technical writing, and oral presentations. However, this reflective writing students do as they look back gives them a chance to be honest and objective. It will also be a great tool to open the following year with—let them be reminded of their goals in their own handwriting!


1. I practiced a lot to be able to present in front of other people.
2. I expanded my paper a lot and went in depth

1. Get work done earlier so I remember what was disscussed in a consultation.
2. Go to more extra consultations in the begining of the year.


Things I did well:
1) Came to extra consultations
2) I did my work after consultations (did not wait till last minute)
Things to improve:
1) Choose a topic sooner
2) talk with more volume

Implementing a Dedicated Math Research Elective

A math research course can be set up as a one-semester high school or college course. A one-semester college course could be set up as a ½ or 1 credit mathematics course, with appropriately spaced class meetings, or as an independent study program with consultations only, scheduled by student and instructor.

A one-semester high school course could meet either every day for one semester or every other day for the entire school year. There are several ways to put in one semester's worth of time over a full school year. If the school has a rotating schedule, the course could meet every "even" day or every "odd" day. Or the research class might meet on days when students have no science lab or no physical education, depending on how your school sets up its master schedule. The longer time spread of the course allows the extra time needed to revise, edit, reformulate conjectures, devise proofs, and so on in between class meetings. If this arrangement is impossible, the course can meet every day for one semester. Although the continuity and concentration levels are high with this format, students might find it hard to complete independent work on a nightly basis. With this arrangement, several days should be allotted for certain assignments. Discuss with your mathematics department chairperson, principal, and guidance counselors how a math research program can be scheduled in your school.

We will discuss a one-year course titled Problem Solving and Math Research and make suggestions for a one-year follow-up course called Investigations in Math Research. The first time your school offers Problem Solving and Math Research, the course must be advertised. As a new course, it could easily be overlooked, even if it appears in the school's course catalog. You can create a flyer about the course and have it distributed by math instructors. Students from all grades can sign up.

Problem Solving and Math Research

The focus in a Problem Solving and Math Research course is on the problem- solving strategies discussed in Chapter 3 and the process of writing research papers. The two topics are treated concurrently. This should be explained at an orientation session, which can be held the first one or two class sessions. Students should receive a copy of *Writing Math Research Papers* and be asked to read the Introduction and Chapter 1 for homework that night. Chapter 1 can be discussed at the next class meeting. Let's look at the problem-solving and research paper components of the one-semester course in detail along with assessment for such a course.

The Problem-Solving Component

Chapter 3 gets students started on the problem-solving facet of their course. Students should read the chapter for homework, and it can be discussed in class the following day. This introductory course in math research requires the instructor to spend most of the class time on problem-solving strategies.

The first problem-solving sessions of the course feature lessons on George Polya's problem-solving steps. You should use Polya's classic, *How to Solve It,* as a reference; most libraries have this book. *Problem-Solving Strategies: Crossing the River with Dogs, and Other Mathematical Adventures,* by Herr and Johnson, is a recommended resource for the problem-solving component of Problem Solving and Math Research. *The Art of Problem Posing,* by Brown and Walter, is also a helpful reference. There are many other resources for problem-solving material listed in Chapters 12 and 13.

A lesson on each strategy should be presented, with examples, questions, extensions, and homework. You can use a combination of lecture, cooperative learning, and developmental and discovery techniques to build each lesson. Before students can be expected to choose strategies, they need to become adept at each one. The initial presentation of each strategy should include several problems that are solved with that particular strategy. Hopefully, this treatment will allow them to begin to associate types of strategies with types of problems to some degree. After all of the strategies have been covered separately, the homework assignments and class problems can mix up all of the strategies and/or combine several of them. Non routine problems of all types can be given. Don't make a habit of reviewing classwork problems at the end of the class meeting—students shouldn't get into the mindset of "We'll go over it at the end of the class;I'll wait for that." Be creative in finding incentives for them to keep trying. Occasionally you can give hints. Never accept blank paper without evidence of the students' scrutinizing and underlining the problem (see Chapter 3). Let them list questions they have about the problem if they can't solve it. Students can solve problems cooperatively, independently, in assigned groups, in groups they choose, or sometimes in a setting of their own choice. Some problems may take several sessions to solve; others may take only a few minutes.

Each student keeps a notebook just as he or she would in any other course. Students can be required to write formal solutions to problems as theyare completed in class. This exercise will help sharpen their writing skills, familiarize them with your editing marks, and give them practice in explaining problem-solving strategies. When the problem-solving component gets under way, look for students who get particularly frustrated with the homework and class problems. These students want to *get the answer*. Becoming used to the frustration and necessary determination that accompany problem solving is not an overnight transition. As the course progresses, the steady diet of problems and their solutions should allow all students to gain confidence. The communication that takes place among students when they are in groups or simply commenting during a class discussion will be very helpful in precipitating success in problem solving.

While the students are solving the problems, you become a roving eavesdropper, asking questions, giving hints and encouragement, and so on. When you review the solutions, you can elicit questions, extensions, and similar problems from the students. Answers are not necessarily given at the end of each class meeting, nor are they necessarily given by the instructor. Your role is to be a discussion stimulator and facilitator. A student can present the solution while you act as a commentator, adding annotations to the student's presentation. At this stage, students' communication skills can be improved by hearing you reword, clarify, and expand upon their explanations. Point out to them that your purpose is to teach them to explain concepts more clearly. As time goes on, ask *them* to reword selected explanations from the problem. Go over homework in a similar fashion. Over a period of months, you should see a dramatic change in the confidence, ability, and zest of your students when they attempt to solve a non routine problem. Frustration, determination, and elation will often go together.

Problem-solving homework assignments can be assigned weekly, handed in, and graded. Each homework assignment can include problems you select, and for high school students, you may want to include questions from old Scholastic Aptitude Tests, Advanced Placement (AP) tests, or other standardized tests. This gives students an early introduction to such tests, helps alleviate the anxiety associated with them, and allows students to become comfortable with the types of questions they will face and with using their problem-solving skills to answer them. You may decide not to give weekly homework when the students begin to write up their research, because this new skill will occupy much of their time.

Where do you find the problems for students to solve? Appendix A lists several books on problem solving. The monthly calendar of *Mathematics Teacher* is a great source of problems. Your math department's library may already have some problem-solving books. Your school or college libraryand public libraries also have books on problem solving. As mentioned in Chapter 2, your classes could create their own problems for a school wide publication. Start a bank of original problems by saving each class's problems. Use these problems in subsequent courses. Some newspapers and magazines feature brainteaser columns; clip these and file them. Virtually all of these problem sources offer solutions, answers, or both. Many sites on the Internet offer Problems of the Week. As you become more comfortable with the problem-solving sessions in your research course, feel free to assign a problem for which you don't have the answer. Students need to see you as an experienced mathematician at work—whether you find a solution or not. The problem-solving component of the course can run through the entire term. The problem-solving experience the students receive should help them clear the hurdles they'll face during their research project.

The Math Research Component

Students will take breaks from their problem solving as they cover chapters in *Writing Math Research Papers*. Use the time line from Chapter 13 to orchestrate the research activities and the reading of each chapter. You can be as flexible with it as you like; it is just a suggestion. As students read each chapter, discuss it in class. Students will complete a Math Annotation Project and a Journal Article Reading Assignment, pick a topic, and get started. Familiarize yourself with the section of Chapters 12 and 13 that have sources for paper topics. You might want to photocopy a selection of articles for students to use. Once the individual research has begun, you can, at any juncture, use class sessions to discuss research material. As the students progress in their research and consultations, assign and discuss the chapters in *Writing Math Research Papers*. The class proceeds through each chapter together, however, the new skills learned manifest themselves in different ways in different papers.

You can allow students to work on their research in class on selected days. On these days, you can conduct consultations with several students during class. When drafts start being handed in, you might want to spend sessions discussing examples of student work that can help the entire class. Toward the end of the course, students can prepare oral presentations and actually present them in class.

Problem Solving and Math Research can be an enriching experience. The articles, the problems, the consultations, the writing, the oral presentations, and your effort and encouragement can transform a student into a budding researcher. If students in Problem Solving and Math Research are also in a core mathematics course, this tandem provides them with a solid, balanced mathematical experience. And if they enjoy Problem Solving and Math Research, they might want to write another paper the following year.

Following Up Your One-Year Program: Investigations in Math Research

After students complete their first mathematics research paper, they can opt to write another on or continue their previous paper in the following school year. In this case, the student will not have a traditional "class" meeting in those subsequent years. Instead, the student participates in a form of independent study, meeting the instructor only for consultations, as frequently as the instructor's schedule permits.

Beginning a Math Research Library

Chapters 12 and 13 list many publications that are appropriate for a math research library. You should try to accumulate as many items as you can by continually adding to your library. A math resource library can be set up as part of the school or college library or as part of the math research classroom or departmental office. In addition to the books and periodicals, a catalog of completed student papers could be set up as a resource for students wishing to *extend* a given paper rather than write their own paper on the same topic. Start your own library of photocopied articles that are good sources for research topics. Search for articles in old journals in libraries. Make lists of readings from resource books that contain potential research topics. Cross-reference the readings and articles that are related. As material accumulates over the years, start a card catalog of research topic ideas for students to use as they search for a topic.

Resource Books

Many of the books in this list are specifically written for instructors, but can also serve as valuable resources for students. Some are general interest books about mathematics, some are about problem solving, and some offer help with research writing. Use Internet booksellers to search for specific books or to look for books on specific topics.

Brown, S., and Walter, M. *The Art of Problem Posing*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1990.

Chazan, D., and Houde, R. *How to Use Conjecturing and Microcomputers to Teach Geometry*. Reston, Va.: NCTM, 1989.

Countryman, J. *Writing to Learn Mathematics*. Portsmouth, N.H.: Heinemann, 1992.

Davis, P., and Hersh, R. *The Mathematical Experience*. Boston: Houghton Mifflin, 1999.

Flegg, G. *Numbers. Their History and Meaning*. New York: Dover, 2002.

Gardner, M. *The Incredible Dr. Matrix. The World's Greatest Numerologist*. New York: Charles Scribner's Sons, 1976.

Guillen, M. *Bridges to Infinity*. Los Angeles: Tarcher, 1983.

Lehoczky, S., and Rusczyk, R. *The Art of Problem Solving—Volume 1: The Basics*. Stanford, Calif.: Greater Testing Concepts, 1993.

Lehoczky, S., and Rusczyk, R. *The Art of Problem Solving—Volume 2: And Beyond*. Stanford, Calif.: Greater Testing Concepts, 1994.

Maletsky, E. *Teaching with Student Math Notes 1*. Reston, Va.: NCTM, 1987.

Maletsky, E. *Teaching with Student Math Notes 2*. Reston, Va.: NCTM, 1993. Mathematical Sciences Education Board and National Research Council.

Reshaping School Mathematics. A Philosophy and Framework for Curriculum. Washington, D.C.: National Academy Press, 1990.

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: NCTM, 1989.

National Council of Teachers of Mathematics. *Professional Standards for Teaching Mathematics*. Reston, Va.: NCTM, 1991.

National Research Council. *Everybody Counts. A Report to the Nation on the Future of Mathematics Education*. Washington, D.C.: National Academy Press, 1989.

Paulos, J. *Innumeracy. Mathematical Illiteracy and Its Consequences*. New York: Hill & Wang, 2001.

Polya, G. *How to Solve It*. Princeton, N.J.: Princeton University Press, 1973.

Polya, G. *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving*. New York: John Wiley and Sons, 1981.

Sachs, L. *Projects to Enrich School Mathematics, Level 3*. Reston, Va.: NCTM,1988.

Schaaf, W. *The High School Mathematics Library*. Reston, Va.: NCTM, 1987.

Steen, L. *On the Shoulders of Giants. New Approaches to Numeracy*. Washington, D.C.: National Academy Press, 2002.

Whimbey, A., and Lochhead, J. *Beyond Problem Solving and Comprehension*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1984.

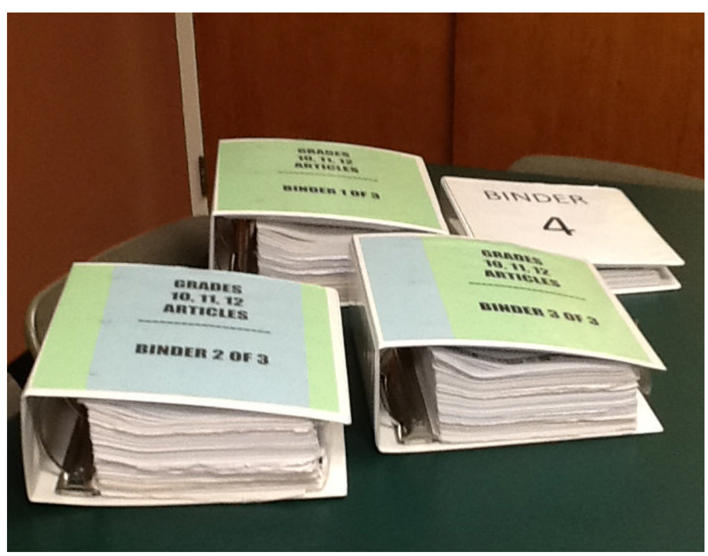
Whimbey, A., and Lochhead, J. *Problem Solving and Comprehension*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1999.

Catalogs of Resource Books

Some catalogs feature books that are excellent resources for research papers. You can send for the catalog or search online and acquire books as your budget allows. In some cases, you may be able to preview the book before purchase. In preparation for your research library, do an Internet search to view online catalogs for the following:

- Annenberg/CPB
- Carolina
- COMAP
- Dover Publications
- ETA hand2mind
- Heinemann
- J. Weston Walch
- Mathematical Association of America
- National Council of Teachers of Mathematics
- Springer-Verlag
- Wadsworth
- W. H. Freeman

You can create your own catalog of topics using the suggested articles, books and journals from chapters 12 and 13. You can find and copy articles that you like, put them in binders, and have kids choose from the binder articles. You can reference textbooks needed for prerequisite material. You can cross reference related articles. The articles can be downloads or copies made from books and journals found in a library.



Students can look through the binders to find a topic and copy any article that they pick. Binders can remain on the shelf of the research library, and used to find topics as well as articles that can help a student extend their research project.

Beyond the Classroom: Math Contests and More!

Some popular mathematics and science contests are listed here. Do an Internet search to find out contact information. Then request rules, regulations and dates from each contest. Keep in mind that most "science" contests include mathematics research. Find out if there are any examples of previous contests that students can use to orient themselves.

- American Mathematics Competitions
- American Regional Math League
- American Scholastic Mathematics Association
- Atlantic-Pacific Mathematics League
- Continental Mathematics League
- Destination-Imagination
- DuPont Science Essay Awards Program
- Google Science Fair
- Harvard/MIT Math Contest
- High School Mathematics Contest in Modeling
- Intel Science Talent Search
- International Mathematical Talent Search
- International Science and Engineering Fair
- Junior Engineering Technical Society
- The Mandelbrot Competition
- Math League Press
- Mathfax Competition
- Moody's Mega Math Challenge
- NASA Space Science Student Involvement Program
- National Junior Science and Humanities Symposium
- National Problem Solving Competition
- Odyssey of the Mind
- Purple Comet
- Rocket City Math Contest
- Siemens-Westinghouse Competition
- The Al Kalfus Long Island Math Fair
- Toshiba NSTA/Exploravision Contest
- USA Mathematical Talent Search

For various reasons, all students may not be able to enter all contests. Perhaps the deadlines conflict with other student commitments. Some only allow seniors. Some require teams. Perhaps the school cannot support a research program or the hiring of a mentor. Maybe the students have other priorities and cannot devote the time and energy needed to enter a competition. In any case, your research students should all have a chance to give their oral presentations in a formal forum.

Several districts and counties have math competitions in place already; schools have math teams that are part of interscholastic math leagues that focus on problem solving. The network of these schools is an excellent starting point for a local math research contest. A committee of interested instructors could meet, set up rules, and create an annual competition. New York has an excellent model for this. In the Al Kalfus Long Island Math Fair, students participate in two rounds: a preliminary round in March or April and a final round in April or May. All students who submit papers are separated according to grade level. The fair includes students in grades 7–12. The students participate in the preliminary round by submitting their papers and giving a fifteen-minute oral presentation to a room with one to three judges and five to eight student presenters. The judges volunteer; they are teachers, businesspeople, professors, and community members. The event is held at a local college. After the students in the room present their papers and answer judges' questions, several students are selected from each room to participate in the second round. At the second round, the preliminary-round procedures are repeated with the selected students. Winners of gold, silver, and bronze medals are chosen from each room and announced at an awards assembly. Specific questions about the fair can be sent to the address listed on the next page. If you are starting a math fair in your community, consider using videotapes of the oral presentations as submissions with the papers if it is impossible to arrange a live event. You may have a student who would like to present a paper at a math department faculty meeting.

In addition to, or in lieu of, a competition, your school or district can hold a Math Research Night at the school. Invite instructors, students, parents, and community businesspeople, and serve refreshments. Artwork from the oral presentations can be on display before and after the presentations. Students can give abbreviated versions of their oral presentations, with a special effort to convey their work in an effective way to people who are not mathematicians. The presentations can be given by one student at a time to a seated audience, or each student can have a table in the gymnasium to present their materials. Attendees can then move around from table to table, asking students about their work. Students are available to speak about their work for the entire evening. The math department could issue awards to students based on criteria established before the event. Money from admission tickets can be used to help defray the costs of the research program, such as books, field trips, materials for students' posters, and so on. The event can be held annually in the spring semester. The school community is used to watching theater events, sports events, and musical events performed by students. Certainly, a student performance displaying the academic power, knowledge, and enthusiasm acquired through a math research project epitomizes the academic mission of the school and deserves to be at center stage.

There are many other ancillary experiences available to the math research students beyond the classroom, as individuals or as a group. Research classes sometimes take day or overnight trips to contests, science museums, math fairs, math teachers' conferences, lectures, and special events. Some colleges allow high school students to shadow college students to get a taste of college classes. In some cases, students can stay overnight in the dormitories. Contact the colleges your students are interested in for details. Students are also welcome, at a reduced registration fee, at NCTM Regional and National meetings. Registered students can attend sessions, and you can require them to report to the class what they learned. A Regional or National Meeting may at some point be held in your area; look in the NCTM website for advance information. Find out if your state or county math teachers' meeting allows students to attend. Student volunteers are usually needed to help out at meetings; your students may be interested in this. If you are requiring students to pay for transportation, meals, admission fees, or hotel costs, be sensitive to their ability to afford the trip you plan. Consider fund-raising activities similar to the fund-raisers held by school clubs. These cold include bake sales, carnivals, contests, benefit concerts, etc.

As you can see, after reading all 13 chapters, a math research program can be a vibrant, stimulating, captivating, colorful, exciting and rewarding endeavor!

Please feel free to contact the author, Dr. Robert Gerver, if you ever have any questions about your program. You can request a PowerPoint presentation if you plan on proposing a math research course to your administration. You can contact Dr. Gerver through the Information Age Publishing website.

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