Diploma Programme

Mathematics mind map

Applications and interpretation AHL

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Mathematics mind map

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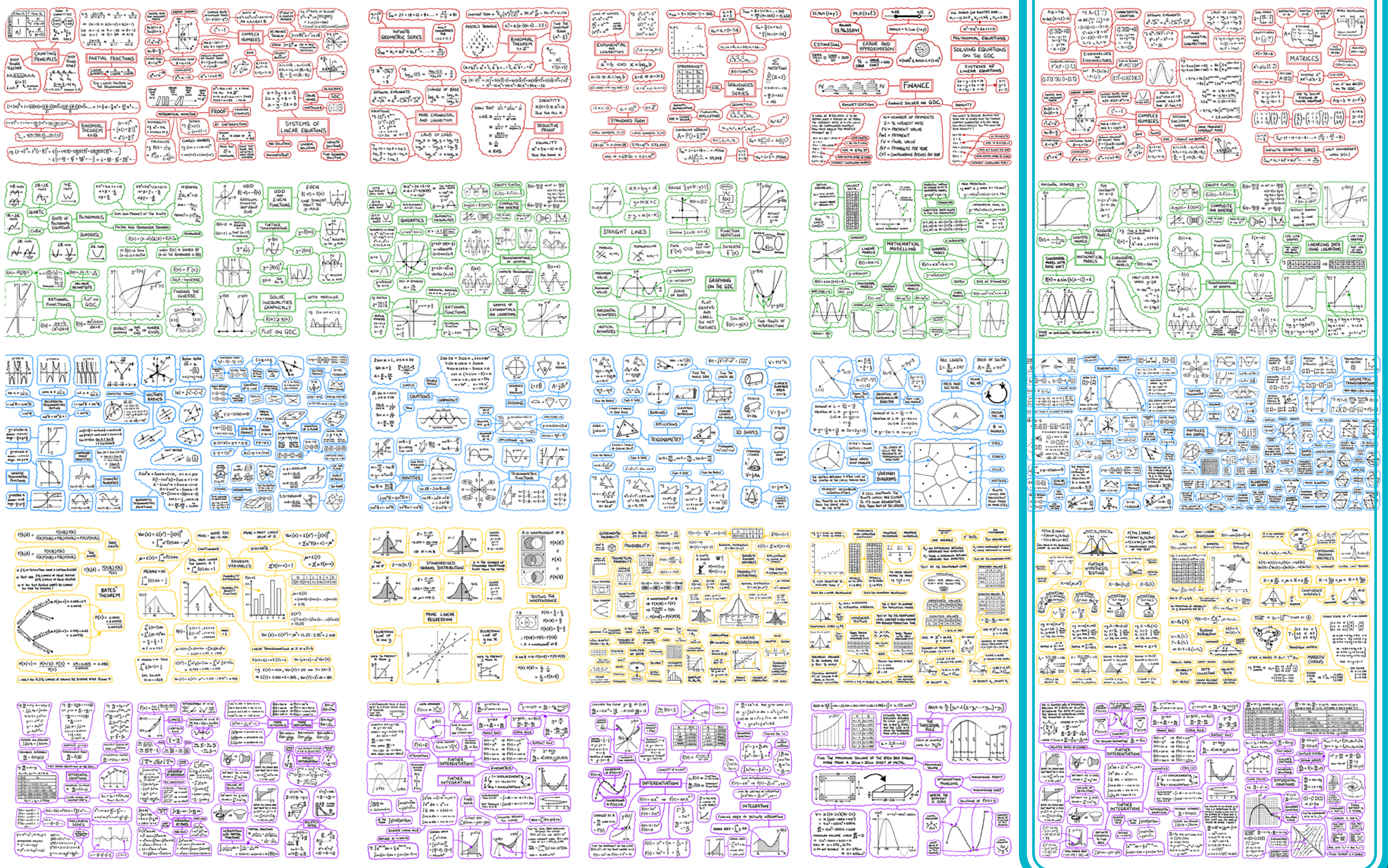
	Analysis and approaches AHL	Analysis and approaches SL	Common content	Applications and interpretation SL	Applications and interpretation AHL
Number and algebra					
Functions					
Geometry and trigonometry					
Statistics and probability					
Calculus					



Applications and interpretation AHL

Number and algebra

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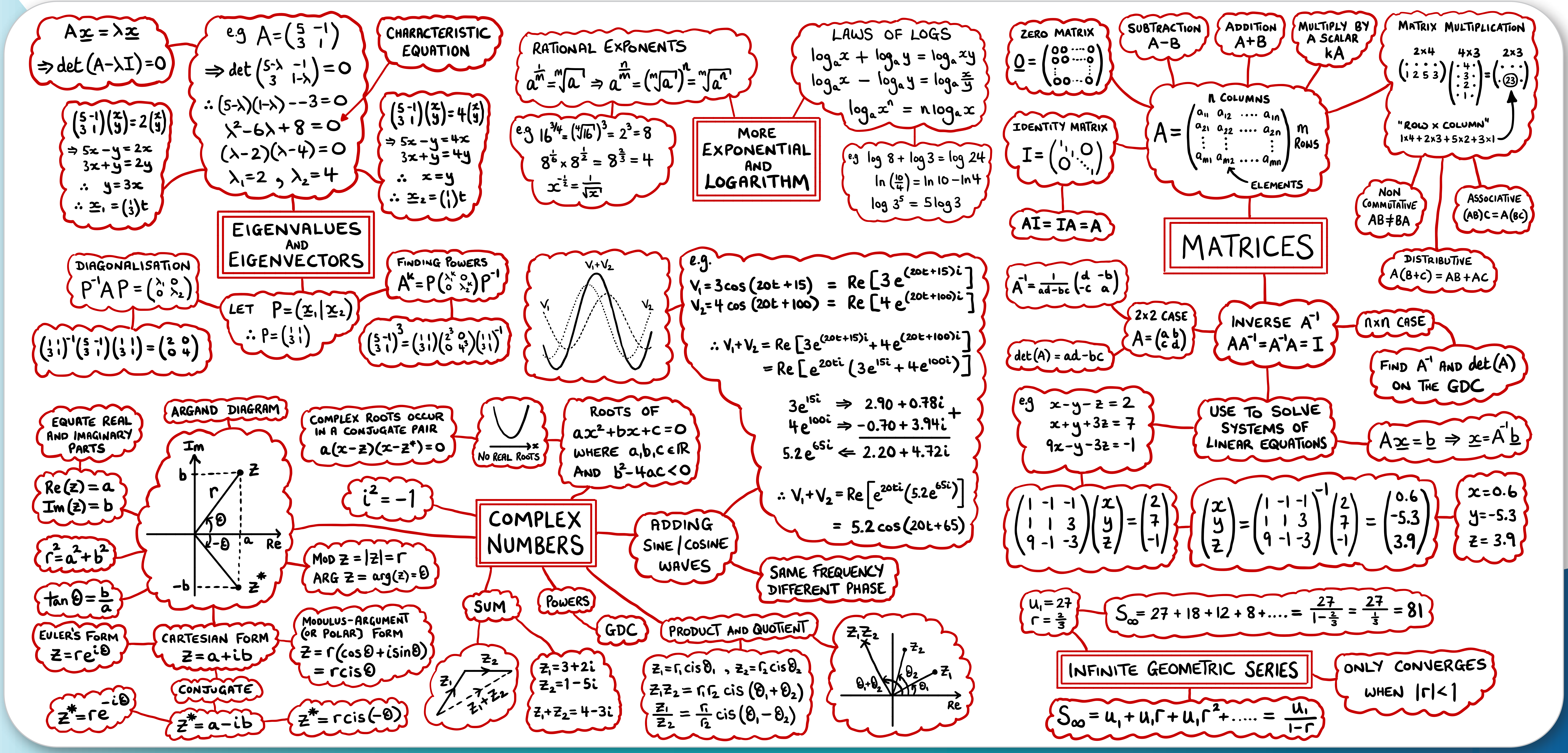
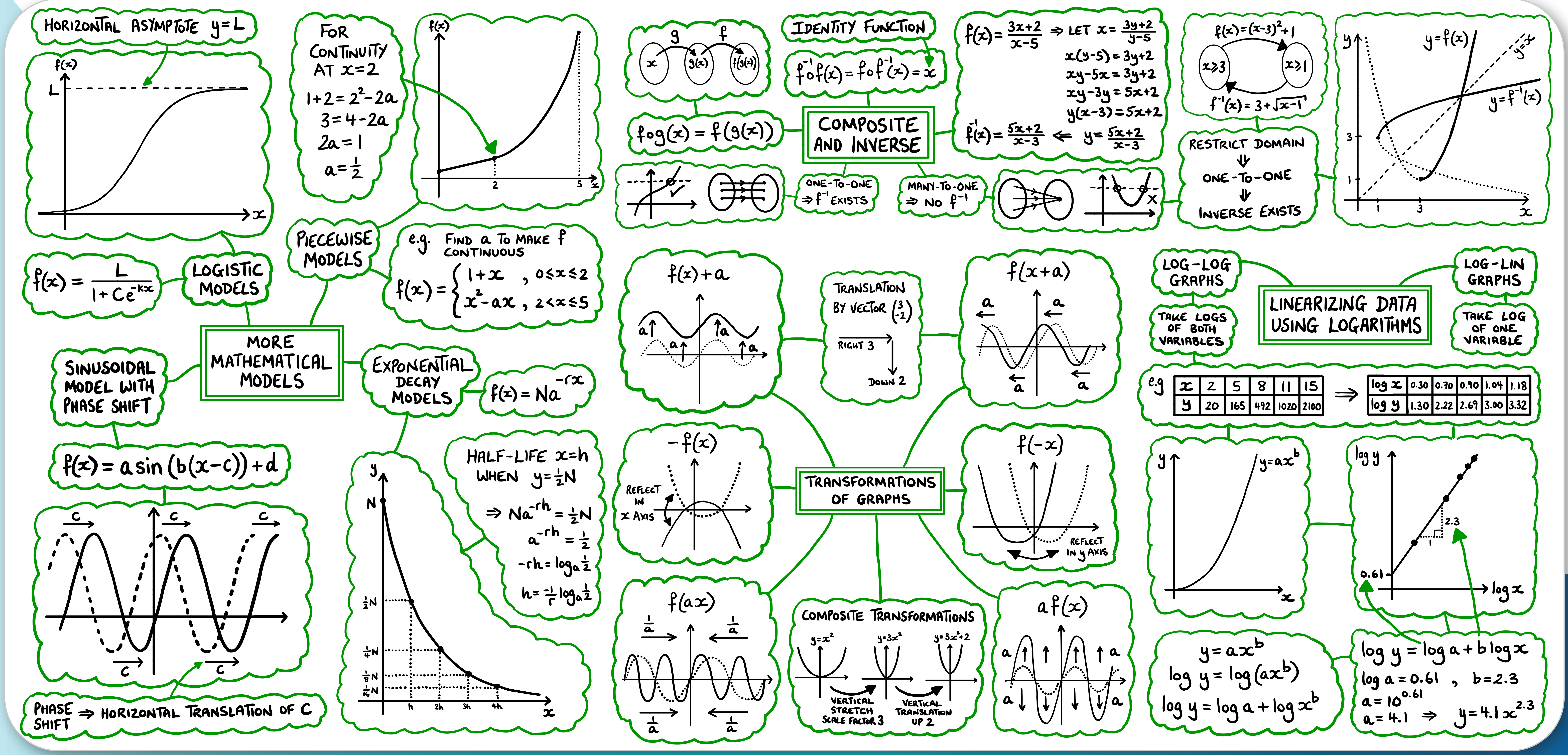


Functions



HORIZONTAL ASYMPTOTE y = L
f(x)
L
x
f(x) = L / (1 + Ce^{-kx})
LOGISTIC MODELS
FOR CONTINUITY AT x = 2
1 + 2 = 2² − 2a
3 = 4 − 2a
2a = 1
a = ½
f(x)
2
5
x
PIECEWISE MODELS
e.g. FIND a TO MAKE f CONTINUOUS
f(x) = { 1 + x , 0 ≤ x ≤ 2 ; x² − ax , 2 < x ≤ 5 }
MORE MATHEMATICAL MODELS
SINUSOIDAL MODEL WITH PHASE SHIFT
f(x) = a sin (b(x − c)) + d
c
c
c
c
c
PHASE SHIFT ⇒ HORIZONTAL TRANSLATION OF C
EXPONENTIAL DECAY MODELS
f(x) = Na^{-rx}
HALF-LIFE x = h WHEN y = ½N
⇒ Na^{-rh} = ½N
a^{-rh} = ½
−rh = log_a ½
h = −(1/r) log_a ½
y
N
½N
¼N
⅛N
1/16 N
h
2h
3h
4h
x
g
f
x
g(x)
f(g(x))
f∘g(x) = f(g(x))
IDENTITY FUNCTION
f⁻¹∘f(x) = f∘f⁻¹(x) = x
COMPOSITE AND INVERSE
ONE-TO-ONE ⇒ f⁻¹ EXISTS
MANY-TO-ONE ⇒ NO f⁻¹
f(x) = (3x + 2)/(x − 5) ⇒ LET x = (3y + 2)/(y − 5)
x(y − 5) = 3y + 2
xy − 5x = 3y + 2
xy − 3y = 5x + 2
y(x − 3) = 5x + 2
f⁻¹(x) = (5x + 2)/(x − 3) ⇐ y = (5x + 2)/(x − 3)
f(x) = (x − 3)² + 1
x ≥ 3
x ≥ 1
f⁻¹(x) = 3 + √(x − 1)
RESTRICT DOMAIN ⇓ ONE-TO-ONE ⇓ INVERSE EXISTS
y
y = f(x)
y = x
y = f⁻¹(x)
3
1
1
3
x
f(x) + a
a
a
a
a
TRANSLATION BY VECTOR (3, −2)
RIGHT 3
DOWN 2
f(x + a)
a
a
a
a
−f(x)
REFLECT IN x AXIS
TRANSFORMATIONS OF GRAPHS
f(−x)
REFLECT IN y AXIS
f(ax)
1/a
1/a
1/a
1/a
COMPOSITE TRANSFORMATIONS
y = x²
y = 3x²
y = 3x² + 2
VERTICAL STRETCH SCALE FACTOR 3
VERTICAL TRANSLATION UP 2
a f(x)
a
a
a
a
LOG-LOG GRAPHS
TAKE LOGS OF BOTH VARIABLES
LINEARIZING DATA USING LOGARITHMS
LOG-LIN GRAPHS
TAKE LOG OF ONE VARIABLE
e.g

x	2	5	8	11	15
y	20	165	492	1020	2100

⇒

log x	0.30	0.70	0.90	1.04	1.18
log y	1.30	2.22	2.69	3.00	3.32

y
y = ax^b
x
log y
2.3
1
0.61
log x
y = ax^b
log y = log (ax^b)
log y = log a + log x^b
log y = log a + b log x
log a = 0.61 , b = 2.3
a = 10^{0.61}
a = 4.1 ⇒ y = 4.1x^{2.3}

Applications and interpretation AHL

Geometry and trigonometry

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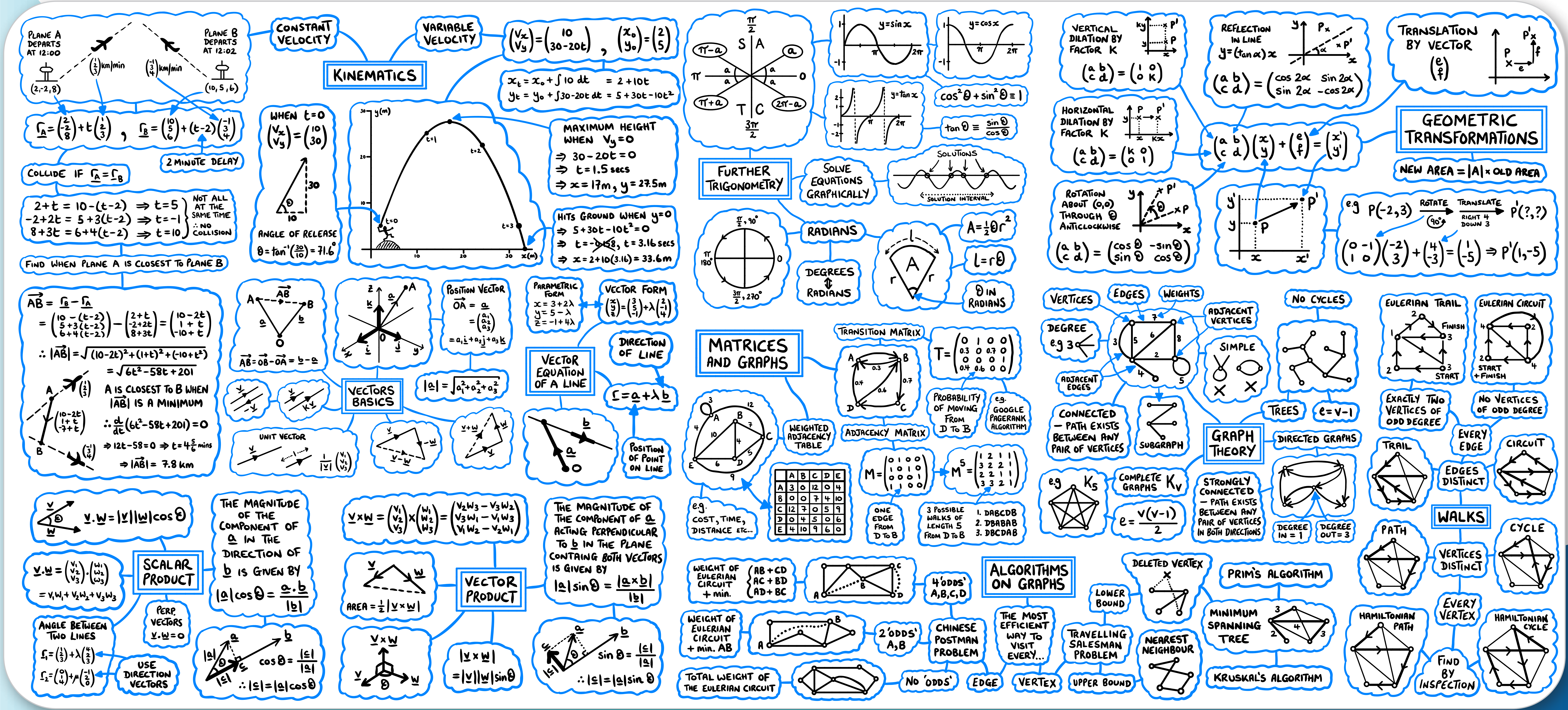

Statistics and probability

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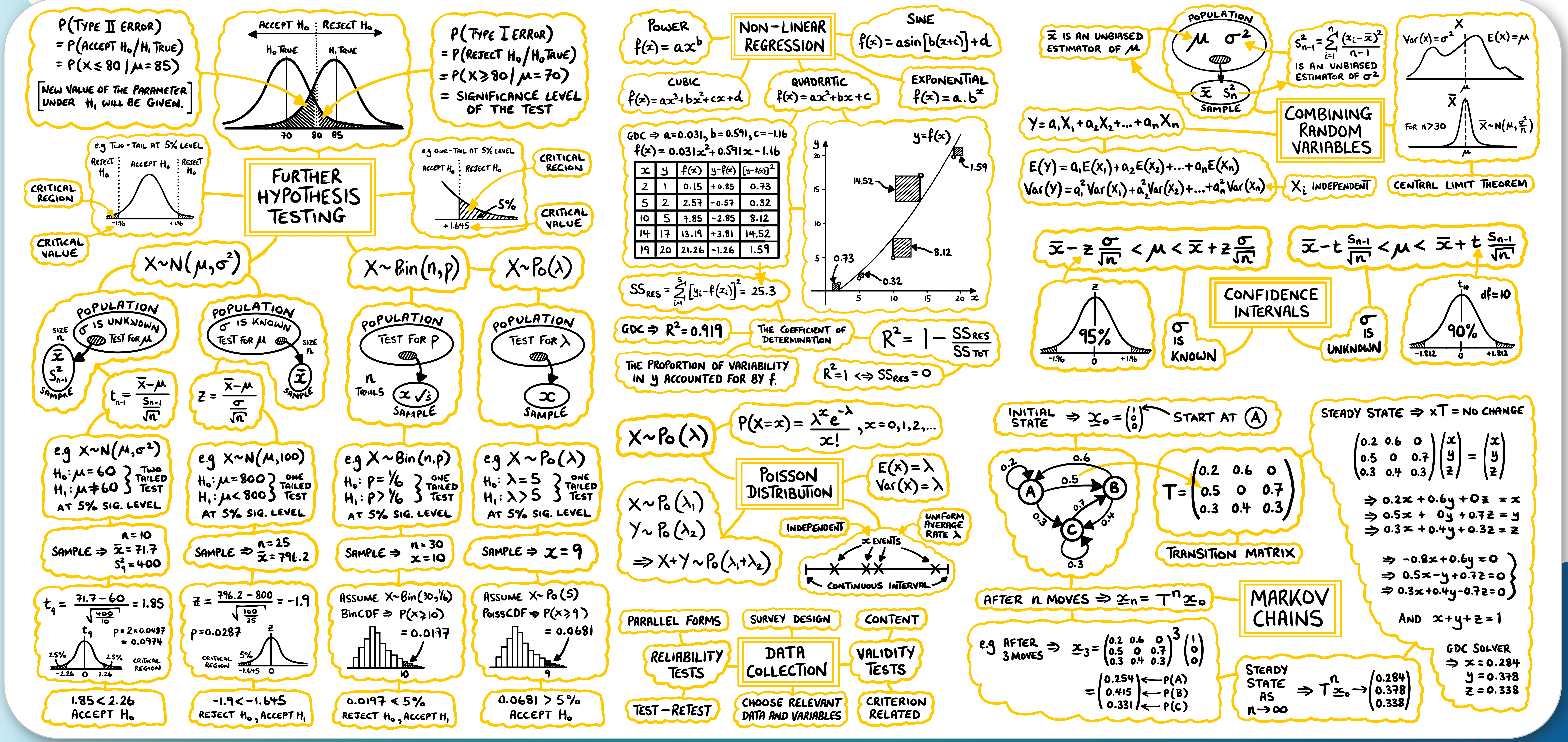
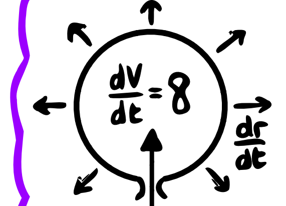


Calculus

Further Differentiation

Related rates of change

Air is pumped into a spherical balloon at a rate of 8 cm³/sec. Determine the rate at which the radius is increasing when the diameter is 10 cm.



Sphere $\Rightarrow V=\frac{4}{3}\pi r^3$

$\therefore \frac{dV}{dr}=4\pi r^2$

$\frac{dr}{dt}=\frac{dr}{dV}\times\frac{dV}{dt}=\frac{8}{4\pi r^2}$

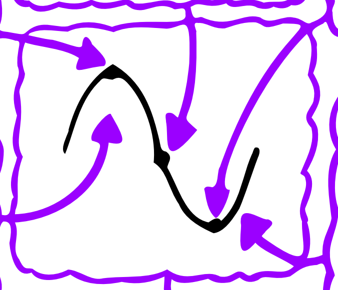
$\therefore$ When $r=5$ cm $\Rightarrow \frac{dr}{dt}=\frac{8}{4\pi .5^2}=\frac{2}{25\pi}$ cm/s

Concavity

Local maximum $f''(x)\leq 0$

Point of inflexion is where the concavity changes

Local minimum $f''(x)\geq 0$



Concave down $f''(x)<0$

Concave up $f''(x)>0$

The second derivative $\frac{d^2y}{dx^2}$ or $f''(x)$

Product rule

e.g. $y=x^2\sin x$

$\frac{dy}{dx}=x^2\cos x+2x\sin x$

$y=uv$

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

Chain rule

e.g. $y=\sin(x^2) \Rightarrow \frac{dy}{dx}=2x\cdot\cos(x^2)$ (diff.)

$y=g(u)$, $u=f(x)$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

Quotient rule

$y=\frac{u}{v}$

$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

e.g. $y=\frac{\sin x}{x^2}$

$\frac{dy}{dx}=\frac{x^2\cos x-2x\cdot\sin x}{x^4}$

Standard derivatives

$f(x)=x^n \Rightarrow f'(x)=nx^{n-1}$

$f(x)=e^x \Rightarrow f'(x)=e^x$

$f(x)=\ln x \Rightarrow f'(x)=\frac{1}{x}$

$f(x)=\sin x \Rightarrow f'(x)=\cos x$

$f(x)=\cos x \Rightarrow f'(x)=-\sin x$

$f(x)=\tan x \Rightarrow f'(x)=\frac{1}{\cos^2 x}$

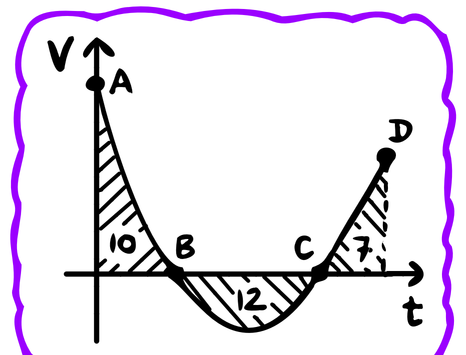
Kinematics

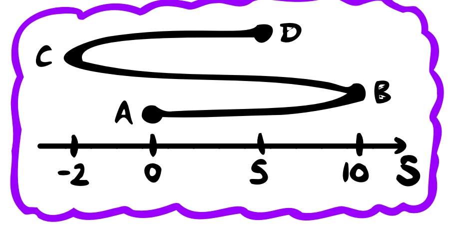
Speed $=|v|$

$\frac{d}{dt}$ S = displacement $\int dt$

V = velocity

$\frac{d}{dt}$ a = acceleration $\int dt$

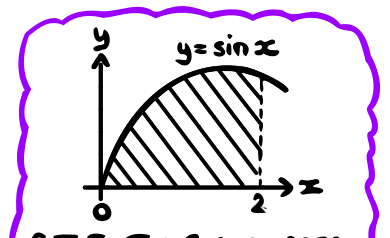




Displacement from A to D $=\int v\,dt=10-12+7=5$ m

Dist. travelled from A to D $=\int|v|\,dt=10+12+7=29$ m

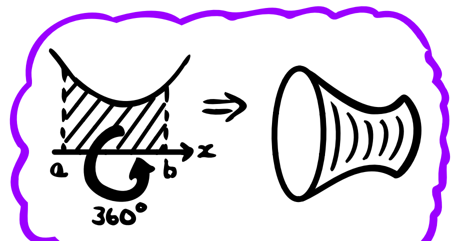
Volume of revolution



Rotate the shaded area 360° about the x-axis

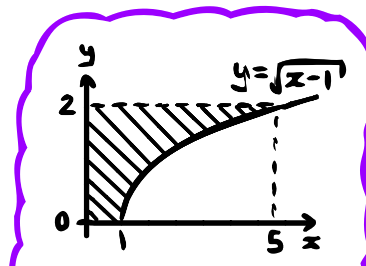
$\therefore V=\pi\int_0^2\sin^2 x\,dx$

$=3.74$ units³



360° about the x-axis

Volume $=\int_a^b \pi y^2\,dx$



Rotate the shaded area 360° about the y-axis

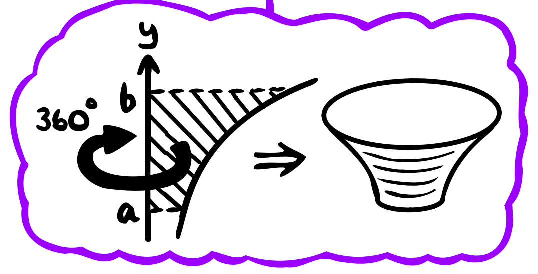
$y=\sqrt{x-1} \Rightarrow y^2+1=x$

$\therefore V=\int_0^2\pi(y^2+1)^2\,dy$

$=43.14$ units³

360° about the y-axis

Volume $=\int_a^b\pi x^2\,dy$



Further Integration

$\int x^n\,dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$

$\int e^x\,dx=e^x+c$

$\int\frac{1}{x}\,dx=\ln|x|+c$

$\int\sin x\,dx=-\cos x+c$

$\int\cos x\,dx=\sin x+c$

$\int\frac{1}{\cos^2 x}\,dx=\tan x+c$

Reverse chain rule

Of the form $\int g'(x)f(g(x))\,dx$

Replace $x\to ax+b$

$\int 2x(x^2+1)^4\,dx=\frac{1}{5}(x^2+1)^5+c$

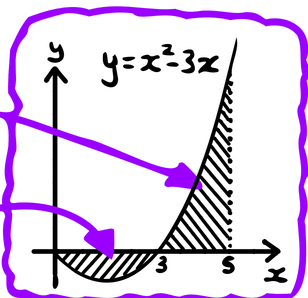
$\int e^{3x+2}\,dx=\frac{1}{3}e^{3x+2}+c$

$\int\cos(2x-1)\,dx=\frac{1}{2}\sin(2x-1)+c$

Definite integration to find area

$\int_a^b g'(x)\,dx=g(b)-g(a)$

$\int_3^5 x^2-3x\,dx$
$=\left[\frac{1}{3}x^3-\frac{3}{2}x^2\right]_3^5$
$=\left(\frac{125}{3}-\frac{75}{2}\right)-\left(9-\frac{27}{2}\right)$
$=8\frac{2}{3}$



$\int_0^3 x^2-3x\,dx$
$=\left[\frac{1}{3}x^3-\frac{3}{2}x^2\right]_0^3$
$=\left(9-\frac{27}{2}\right)-(0)$
$=-4\frac{1}{2}$

Total shaded area $=4\frac{1}{2}+8\frac{2}{3}=13\frac{1}{6}$ units²

Differential equations

Variables separable $\frac{dy}{dx}=f(x)g(y)$

Separate and integrate $\int\frac{1}{g(y)}\,dy=\int f(x)\,dx$

$\frac{dy}{dx}=3y$ and $y=2$ when $x=0$
$\Rightarrow \int\frac{1}{y}\,dy=\int 3\,dx$
$\Rightarrow \ln|y|=3x+c$
$\Rightarrow y=e^{3x+c}=e^{3x}.e^c=Ae^{3x}$
$(0,2)\Rightarrow 2=Ae^{3(0)} \Rightarrow A=2$
$\therefore y=2e^{3x}$

The growth of an algae G at time t is proportional to $\sqrt{G}$. Initially algae covers 9 m² of the surface of a pond and 4 days later this increases to 25 m². What will the area of algae be after 1 week?

$\frac{dG}{dt}=k\sqrt{G} \Rightarrow \int\frac{1}{\sqrt{G}}\,dG=\int k\,dt$

$\therefore 2G^{1/2}=kt+c$

$t=0, G=9 \Rightarrow c=6$

$\therefore 2G^{1/2}=kt+6$

$t=4, G=25 \Rightarrow k=1$

$\therefore G=\left(\frac{t+6}{2}\right)^2$

$t=7 \Rightarrow G=\left(\frac{7+6}{2}\right)^2=42.25$ m²

Differential Equations

Numerical solutions — Euler's method

$\frac{dy}{dx}=x+y$. When $x=0, y=1$

Using $h=0.2$ estimate y when $x=1$

n	x_n	$y_n \Rightarrow$ store $0.2\to h$, use ANS on GDC
0	0	$=1$
1	0.2	$=$ ANS $+h(0+$ ANS$)=1.2$
2	0.4	$=$ ANS $+h(0.2+$ ANS$)=1.48$
3	0.6	$=$ ANS $+h(0.4+$ ANS$)=1.856$
4	0.8	$=$ ANS $+h(0.6+$ ANS$)=2.347$
5	1	$=$ ANS $+h(0.8+$ ANS$)=2.977$

$x_5=1$, $y_5=2.977$

$\frac{dx}{dt}=3x-y$, $\frac{dy}{dt}=x+y+t$. When $t=0, x=2, y=1$

Using a step of $h=\frac{1}{4}$, estimate x and y when $t=1$

t	x	y
0	$x_0=2$	$y_0=1$
0.25	$x_1=2+(3\times2-1)(\frac{1}{4})=3.25$	$y_1=1+(2+1+0)(\frac{1}{4})=1.75$
0.5	$x_2=3.25+(3\times3.25-1.75)(\frac{1}{4})=5.25$	$y_2=1.75+(3.25+1.75+0.25)(\frac{1}{4})=3.063$
0.75	$x_3=5.25+(3\times5.25-3.063)(\frac{1}{4})=8.422$	$y_3=3.063+(5.25+3.063+0.5)(\frac{1}{4})=5.266$
1	$x_4=8.422+(3\times8.422-5.266)(\frac{1}{4})=13.422$	$y_4=5.266+(8.422+5.266+0.75)(\frac{1}{4})=8.875$

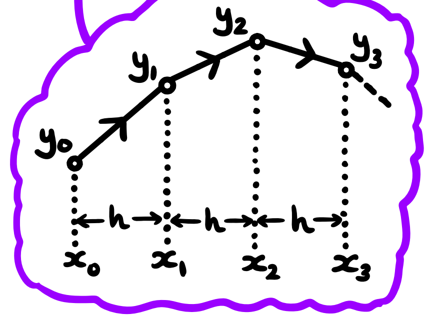
$\therefore$ When $t=1 \Rightarrow x=13.422$, $y=8.875$

First order

$y_{n+1}=y_n+hf(x_n,y_n)$

$x_{n+1}=x_n+h$

$\frac{dy}{dx}=f(x,y)$



Coupled system

$\frac{dx}{dt}=f_1(x,y,t)$

$\frac{dy}{dt}=f_2(x,y,t)$

$x_{n+1}=x_n+f_1(x_n,y_n,t_n)h$

$y_{n+1}=y_n+f_2(x_n,y_n,t_n)h$

Second order

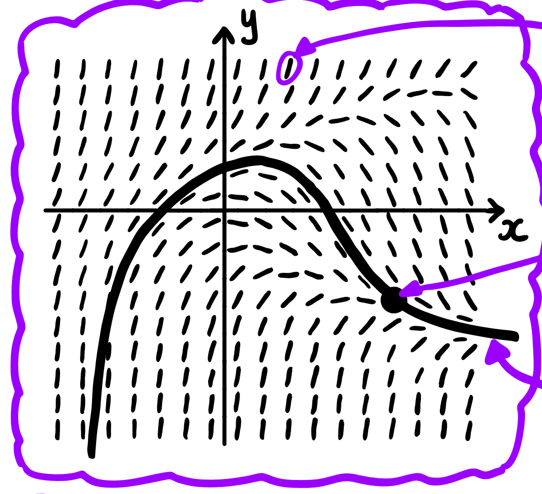
$\frac{d^2x}{dt^2}=f(x,\frac{dx}{dt},t)$

Let $y=\frac{dx}{dt} \Rightarrow \frac{dy}{dt}=\frac{d^2x}{dt^2}$

Now solve $\begin{cases}\frac{dx}{dt}=y\\ \frac{dy}{dt}=f(x,y,t)\end{cases}$

Slope fields

e.g. $\frac{dy}{dx}=y^2-x$



At (1,2) $\frac{dy}{dx}=2^2-1=3$

Boundary condition

Solution curve

Coupled systems

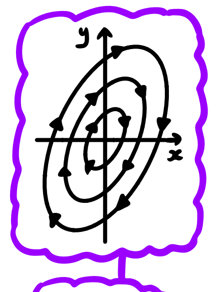
$\frac{dx}{dt}=ax+by$

$\frac{dy}{dt}=cx+dy$

$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$

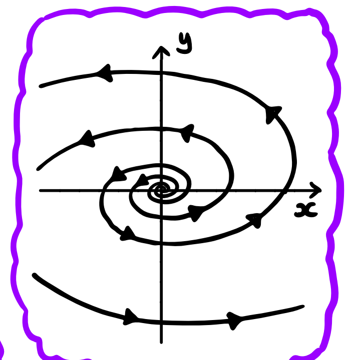
$\Rightarrow \dot{\mathbf{x}}=A\mathbf{x}$

Find the eigenvalues of A $\Rightarrow \lambda_1,\lambda_2$



Phase portrait is a circle or ellipse

Imaginary λ_1 and λ_2



Phase portrait is a spiral

Complex λ_1 and λ_2

Real part of both λ_1 and λ_2 is $-$ — solutions move towards the origin

Real part of both λ_1 and λ_2 is $+$ — solutions move away from origin

Investigate the phase portrait... $\frac{dx}{dt}=x+2y$, $\frac{dy}{dt}=3x+2y$

$\Rightarrow \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}1 & 2\\ 3 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$

$\begin{vmatrix}1-\lambda & 2\\ 3 & 2-\lambda\end{vmatrix}=0$

$\lambda^2-3\lambda-4=0$

$(\lambda-4)(\lambda+1)=0$

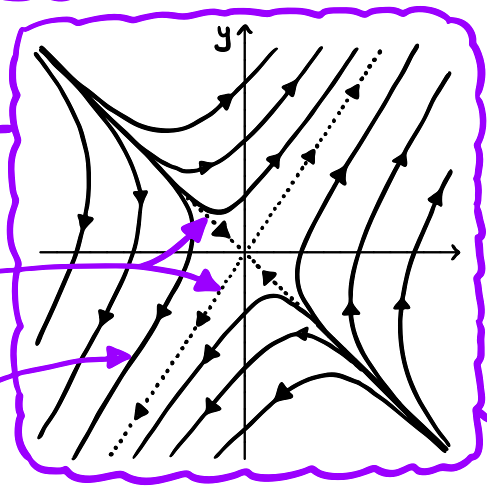
$\lambda_1=4, \lambda_2=-1$

Eigenvectors

$\lambda_1=4 \Rightarrow \mathbf{x}_1=\begin{pmatrix}2\\ 3\end{pmatrix}$

$\lambda_2=-1 \Rightarrow \mathbf{x}_2=\begin{pmatrix}1\\ -1\end{pmatrix}$

Different signs $\Rightarrow$ solutions move away from origin



Real with $\lambda_1=+$ and $\lambda_2=-$

Phase portrait is a saddle