

# Mathematics mind map

## Applications and interpretation AHL

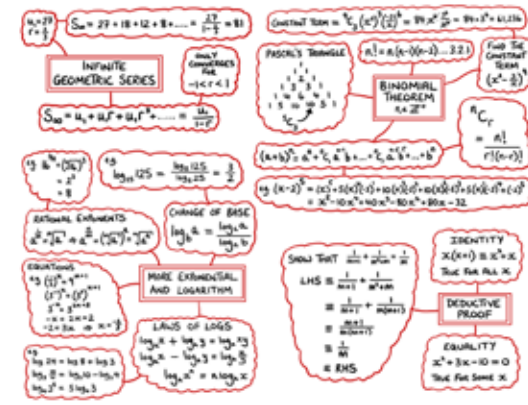
# Mathematics mind map

## Number and algebra

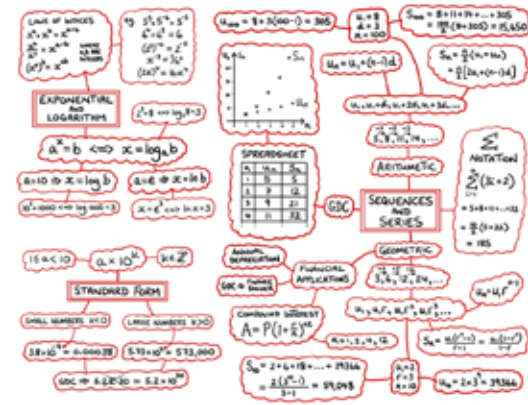
### Analysis and approaches AHL



### Analysis and approaches SL



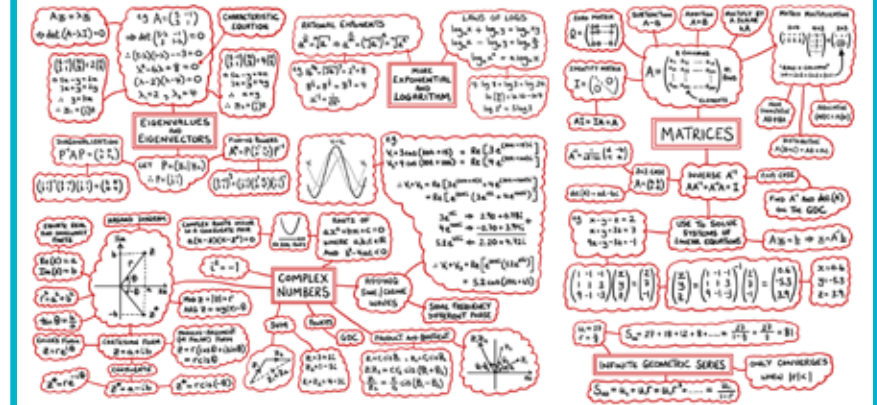
### Common content



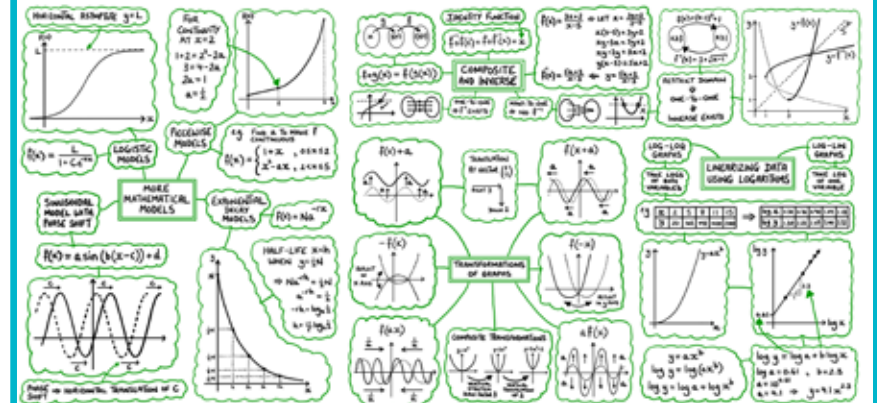
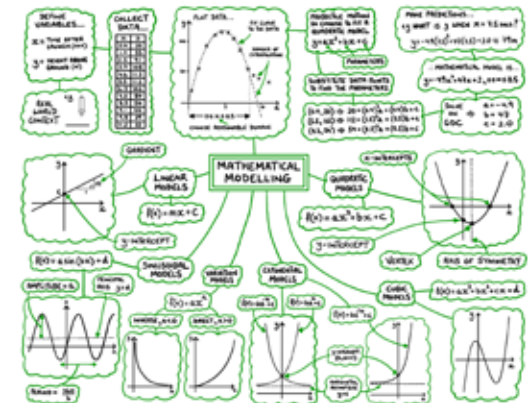
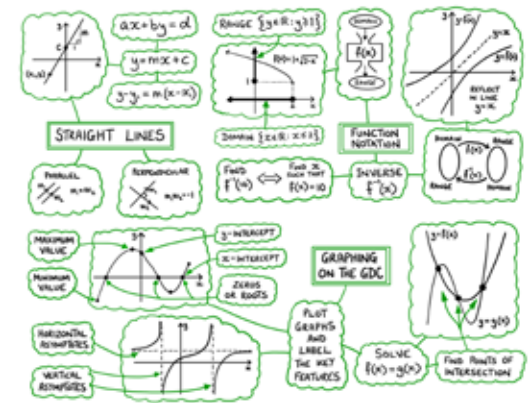
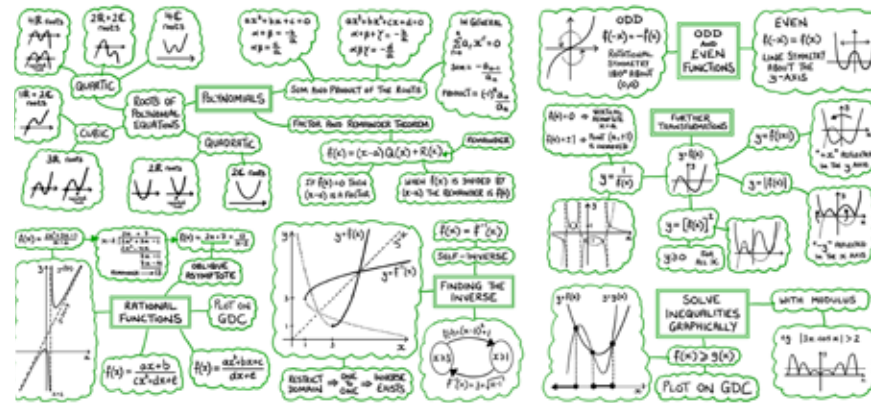
### Applications and interpretation SL



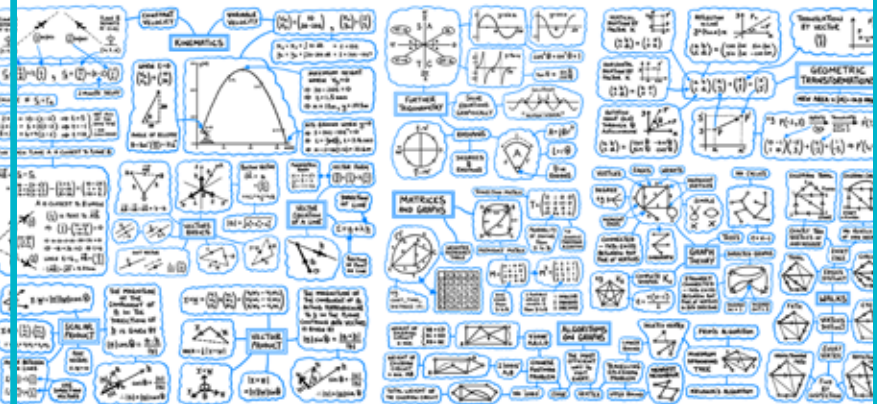
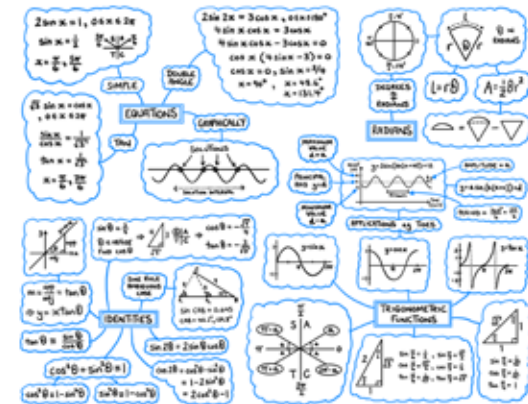
### Applications and interpretation AHL



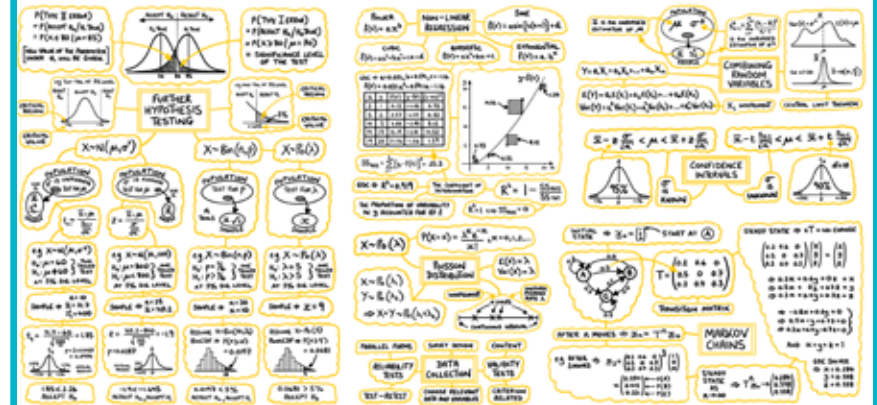
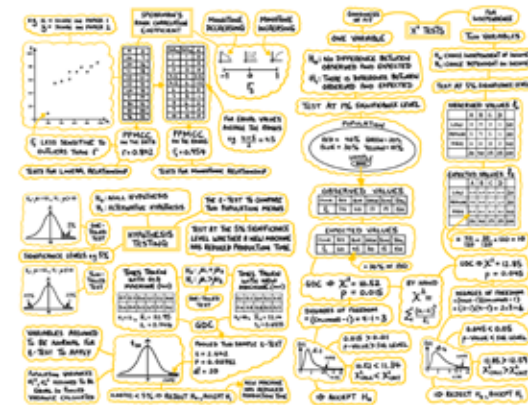
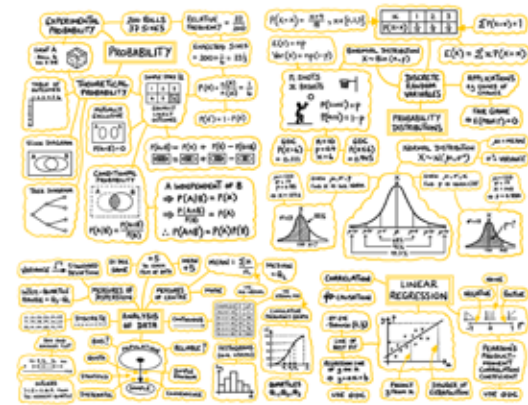
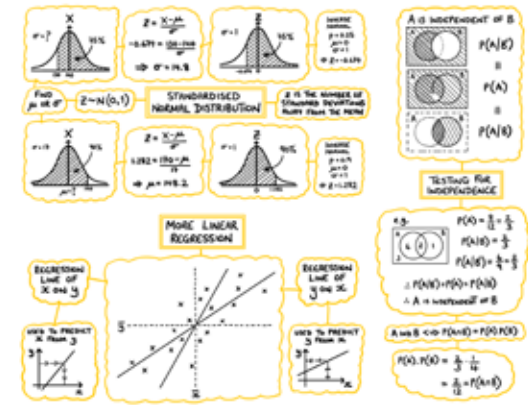
## Functions



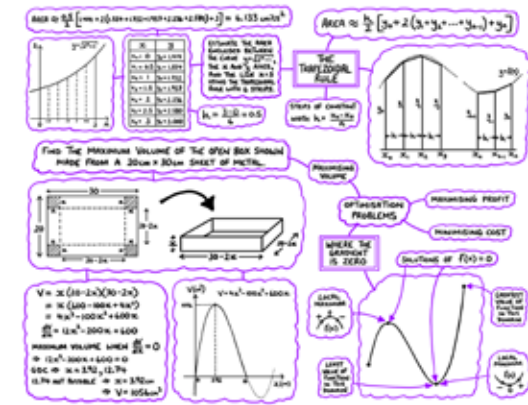
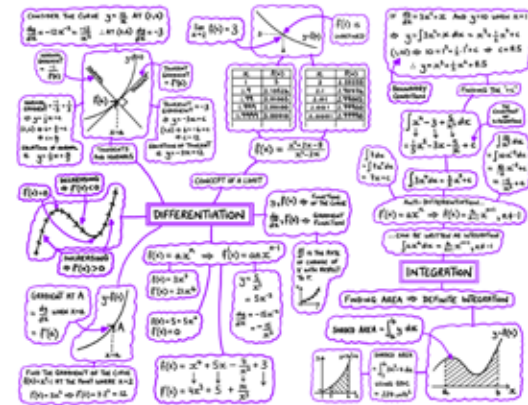
## Geometry and trigonometry



## Statistics and probability



## Calculus





**HORIZONTAL ASYMPTOTE  $y=L$**

**FOR CONTINUITY AT  $x=2$**

$$1+2 = 2^2 - 2a$$

$$3 = 4 - 2a$$

$$2a = 1$$

$$a = \frac{1}{2}$$

**IDENTITY FUNCTION**

$$f^{-1} \circ f(x) = f \circ f^{-1}(x) = x$$

**COMPOSITE AND INVERSE**

ONE-TO-ONE  $\Rightarrow f^{-1}$  EXISTS

MANY-TO-ONE  $\Rightarrow$  NO  $f^{-1}$

**COMPOSITE FUNCTIONS**

$$f \circ g(x) = f(g(x))$$

**INVERSE FUNCTIONS**

$$f(x) = \frac{3x+2}{x-5} \Rightarrow \text{LET } x = \frac{3y+2}{y-5}$$

$$x(y-5) = 3y+2$$

$$xy - 5x = 3y+2$$

$$xy - 3y = 5x+2$$

$$y(x-3) = 5x+2$$

$$f^{-1}(x) = \frac{5x+2}{x-3} \leftarrow y = \frac{5x+2}{x-3}$$

**RESTRICT DOMAIN**

ONE-TO-ONE

INVERSE EXISTS

**PIECEWISE MODELS**

e.g. FIND  $a$  TO MAKE  $f$  CONTINUOUS

$$f(x) = \begin{cases} 1+x, & 0 \leq x \leq 2 \\ x^2 - ax, & 2 < x \leq 5 \end{cases}$$

**LOGISTIC MODELS**

$$f(x) = \frac{L}{1 + Ce^{-kx}}$$

**SINUSOIDAL MODEL WITH PHASE SHIFT**

$$f(x) = a \sin(b(x-c)) + d$$

**EXPONENTIAL DECAY MODELS**

$$f(x) = Na^{-rx}$$

**TRANSFORMATIONS OF GRAPHS**

TRANSLATION BY VECTOR  $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

RIGHT 3

DOWN 2

**LOG-LOG GRAPHS**

TAKE LOGS OF BOTH VARIABLES

**LOG-LIN GRAPHS**

TAKE LOG OF ONE VARIABLE

**LINEARIZING DATA USING LOGARITHMS**

$x$	2	5	8	11	15
$y$	20	165	492	1020	2100

$\log x$	0.30	0.70	0.90	1.04	1.18
$\log y$	1.30	2.22	2.69	3.00	3.32

**HALF-LIFE  $x=h$  WHEN  $y = \frac{1}{2}N$**

$$\Rightarrow Na^{-rh} = \frac{1}{2}N$$

$$a^{-rh} = \frac{1}{2}$$

$$-rh = \log_a \frac{1}{2}$$

$$h = \frac{1}{r} \log_a \frac{1}{2}$$

**COMPOSITE TRANSFORMATIONS**

$y = x^2$   $\rightarrow$   $y = 3x^2$   $\rightarrow$   $y = 3x^2 + 2$

VERTICAL STRETCH SCALE FACTOR 3

VERTICAL TRANSLATION UP 2

**PHASE SHIFT  $\Rightarrow$  HORIZONTAL TRANSLATION OF  $C$**

**TRANSFORMATIONS OF GRAPHS**

$f(x+a)$

$f(x)+a$

$f(-x)$

$-f(x)$

$f(ax)$

$af(x)$

**LOG-LOG GRAPHS**

**LOG-LIN GRAPHS**

$y = ax^b$

$\log y = \log(ax^b)$

$\log y = \log a + b \log x$

$\log a = 0.61$ ,  $b = 2.3$

$a = 10^{0.61}$

$a = 4.1 \Rightarrow y = 4.1x^{2.3}$

### KINEMATICS

**CONSTANT VELOCITY**

PLANE A DEPARTS AT 12:00  
 (2, -2, 8)  $\left(\frac{2}{3}\right)$  km/min

PLANE B DEPARTS AT 12:02  
 (10, 5, 6)  $\left(\frac{3}{4}\right)$  km/min

$\vec{r}_A = \begin{pmatrix} 2 \\ -2 \\ 8 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ ,  $\vec{r}_B = \begin{pmatrix} 10 \\ 5 \\ 6 \end{pmatrix} + (t-2) \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$

COLLIDE IF  $\vec{r}_A = \vec{r}_B$

2 MINUTE DELAY

2 + t = 10 - (t-2)  $\Rightarrow t = 5$   
 -2 + 2t = 5 + 3(t-2)  $\Rightarrow t = -1$   
 8 + 3t = 6 + 4(t-2)  $\Rightarrow t = 10$

NOT ALL AT THE SAME TIME  
 $\therefore$  NO COLLISION

FIND WHEN PLANE A IS CLOSEST TO PLANE B

$\vec{AB} = \vec{r}_B - \vec{r}_A$   
 $= \begin{pmatrix} 10 - (2+t) \\ 5 + 3(t-2) \\ 6 + 4(t-2) \end{pmatrix} = \begin{pmatrix} 8-2t \\ 5+3t-6 \\ 6+4t-8 \end{pmatrix} = \begin{pmatrix} 8-2t \\ -1+3t \\ -2+4t \end{pmatrix}$

$|\vec{AB}| = \sqrt{(8-2t)^2 + (-1+3t)^2 + (-2+4t)^2}$   
 $= \sqrt{6t^2 - 58t + 201}$

A IS CLOSEST TO B WHEN  $|\vec{AB}|$  IS A MINIMUM

$\frac{d}{dt}(6t^2 - 58t + 201) = 0$   
 $\Rightarrow 12t - 58 = 0 \Rightarrow t = 4\frac{5}{6}$  mins

$\Rightarrow |\vec{AB}| = 7.8$  km

### FURTHER TRIGONOMETRY

**VARIABLE VELOCITY**

$(v_x) = \begin{pmatrix} 10 \\ 30-20t \end{pmatrix}$ ,  $(x_0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$x_t = x_0 + \int 10 dt = 2 + 10t$   
 $y_t = y_0 + \int 30-20t dt = 5 + 30t - 10t^2$

MAXIMUM HEIGHT WHEN  $v_y = 0$   
 $\Rightarrow 30 - 20t = 0 \Rightarrow t = 1.5$  secs  
 $\Rightarrow x = 17.5$  m,  $y = 27.5$  m

HITS GROUND WHEN  $y = 0$   
 $\Rightarrow 5 + 30t - 10t^2 = 0$   
 $\Rightarrow t = -0.158$ ,  $t = 3.16$  secs  
 $\Rightarrow x = 2 + 10(3.16) = 33.6$  m

ANGLE OF RELEASE  $\theta = \tan^{-1}\left(\frac{30}{10}\right) = 71.6^\circ$

**VERTICES**

DEGREE e.g. 3

ADJACENT EDGES

CONNECTED - PATH EXISTS BETWEEN ANY PAIR OF VERTICES

eg  $K_5$

COMPLETE GRAPHS  $K_n$   
 $e = \frac{v(v-1)}{2}$

STRONGLY CONNECTED - PATH EXISTS BETWEEN ANY PAIR OF VERTICES IN BOTH DIRECTIONS

DEGREE IN = 1, DEGREE OUT = 3

WALKS

EVERY EDGE

EDGES DISTINCT

WALKS

VERTICES DISTINCT

EVERY VERTEX

FIND BY INSPECTION

### GEOMETRIC TRANSFORMATIONS

**REFLECTION IN LINE**

$y = (\tan \alpha)x$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$

**TRANSLATION BY VECTOR**

$\begin{pmatrix} e \\ f \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

NEW AREA =  $|A| \times$  OLD AREA

eg  $P(-2, 3)$  ROTATE  $90^\circ$  TRANSLATE RIGHT 4 DOWN 3  $\Rightarrow P'(?, ?)$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix} \Rightarrow P'(1, -5)$

### MATRICES AND GRAPHS

**RADIANS**

$A = \frac{1}{2} \theta r^2$   
 $L = r\theta$

$\theta$  IN RADIANS

**TRANSITION MATRIX**

$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 1 \\ 0.4 & 0.6 & 0 & 0 \end{pmatrix}$

PROBABILITY OF MOVING FROM D TO B

eg. GOOGLE PAGERANK ALGORITHM

**WEIGHTED ADJACENCY TABLE**

	A	B	C	D	E
A	3	0	12	0	4
B	0	0	7	4	10
C	12	7	0	5	9
D	0	4	5	0	6
E	4	10	9	6	0

ONE EDGE FROM D TO B

3 POSSIBLE WALKS OF LENGTH 5 FROM D TO B

1. DABCD B  
 2. DBAB B  
 3. DBCD B

**ALGORITHMS ON GRAPHS**

WEIGHT OF EULERIAN CIRCUIT + min. AB

WEIGHT OF EULERIAN CIRCUIT + min. (AB+CD, AC+BD, AD+BC)

4 'ODDS' A, B, C, D

2 'ODDS' A, B

NO 'ODDS' EDGE

THE MOST EFFICIENT WAY TO VISIT EVERY...  
 TRAVELLING SALESMAN PROBLEM

LOWER BOUND

DELETED VERTEX

NEAREST NEIGHBOUR

PRIM'S ALGORITHM

MINIMUM SPANNING TREE

KRUSKAL'S ALGORITHM

HAMILTONIAN PATH

HAMILTONIAN CYCLE

### VECTORS BASICS

**POSITION VECTOR**

$\vec{OA} = \vec{a}$   
 $= a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

**VECTOR EQUATION OF A LINE**

$\vec{r} = \vec{a} + \lambda \vec{b}$

**SCALAR PRODUCT**

$\vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \theta$

THE MAGNITUDE OF THE COMPONENT OF  $\vec{a}$  IN THE DIRECTION OF  $\vec{b}$  IS GIVEN BY  $|\vec{a}| \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

**VECTOR PRODUCT**

$\vec{v} \times \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$

THE MAGNITUDE OF THE COMPONENT OF  $\vec{a}$  ACTING PERPENDICULAR TO  $\vec{b}$  IN THE PLANE CONTAINING BOTH VECTORS IS GIVEN BY  $|\vec{a}| \sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{b}|}$

AREA =  $\frac{1}{2} |\vec{v} \times \vec{w}|$

$|\vec{v} \times \vec{w}| = |\vec{v}| |\vec{w}| \sin \theta$

**P(TYPE II ERROR)**  
 $= P(\text{ACCEPT } H_0 / H_1 \text{ TRUE})$   
 $= P(X \leq 80 | \mu = 85)$   
 [NEW VALUE OF THE PARAMETER UNDER  $H_1$  WILL BE GIVEN.]

**P(TYPE I ERROR)**  
 $= P(\text{REJECT } H_0 / H_0 \text{ TRUE})$   
 $= P(X \geq 80 | \mu = 70)$   
 = SIGNIFICANCE LEVEL OF THE TEST

**FURTHER HYPOTHESIS TESTING**

e.g. TWO-TAIL AT 5% LEVEL  
 CRITICAL REGION: REJECT  $H_0$  (left of -1.96), REJECT  $H_0$  (right of +1.96)  
 CRITICAL VALUE: ±1.96

e.g. ONE-TAIL AT 5% LEVEL  
 CRITICAL REGION: REJECT  $H_0$  (right of +1.645)  
 CRITICAL VALUE: +1.645

**$X \sim N(\mu, \sigma^2)$**

POPULATION  $\sigma$  IS UNKNOWN TEST FOR  $\mu$   
 SIZE  $n$  SAMPLE  $\bar{x}, s_{n-1}^2$   
 $t_{n-1} = \frac{\bar{x} - \mu}{\frac{s_{n-1}}{\sqrt{n}}}$

POPULATION  $\sigma$  IS KNOWN TEST FOR  $\mu$   
 SIZE  $n$  SAMPLE  $\bar{x}$   
 $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$

**$X \sim \text{Bin}(n, p)$**

POPULATION TEST FOR  $p$   
 $n$  TRIALS SAMPLE  $x$ 's

**$X \sim \text{Po}(\lambda)$**

POPULATION TEST FOR  $\lambda$   
 SAMPLE  $x$

e.g.  $X \sim N(\mu, \sigma^2)$   
 $H_0: \mu = 60$  } TWO TAILED TEST  
 $H_1: \mu \neq 60$  } AT 5% SIG. LEVEL  
 $n = 10$   
 SAMPLE  $\Rightarrow \bar{x} = 71.7, s_{9}^2 = 400$   
 $t_9 = \frac{71.7 - 60}{\sqrt{\frac{400}{10}}} = 1.85$   
 $p = 2 \times 0.0487 = 0.0974$   
 $1.85 < 2.26$  ACCEPT  $H_0$

e.g.  $X \sim N(\mu, \sigma^2)$   
 $H_0: \mu = 800$  } ONE TAILED TEST  
 $H_1: \mu < 800$  } AT 5% SIG. LEVEL  
 $n = 25$   
 SAMPLE  $\Rightarrow \bar{x} = 796.2$   
 $z = \frac{796.2 - 800}{\sqrt{\frac{100}{25}}} = -1.9$   
 $p = 0.0287$   
 $-1.9 < -1.645$  REJECT  $H_0$ , ACCEPT  $H_1$

e.g.  $X \sim \text{Bin}(n, p)$   
 $H_0: p = 1/6$  } ONE TAILED TEST  
 $H_1: p > 1/6$  } AT 5% SIG. LEVEL  
 $n = 30$   
 SAMPLE  $\Rightarrow x = 10$   
 ASSUME  $X \sim \text{Bin}(30, 1/6)$   
 BinCDF  $\Rightarrow P(X \geq 10) = 0.0197$   
 $0.0197 < 5\%$  REJECT  $H_0$ , ACCEPT  $H_1$

e.g.  $X \sim \text{Po}(\lambda)$   
 $H_0: \lambda = 5$  } ONE TAILED TEST  
 $H_1: \lambda > 5$  } AT 5% SIG. LEVEL  
 SAMPLE  $\Rightarrow x = 9$   
 ASSUME  $X \sim \text{Po}(5)$   
 PoissCDF  $\Rightarrow P(X \geq 9) = 0.0681$   
 $0.0681 > 5\%$  ACCEPT  $H_0$

**POWER**  
 $f(x) = ax^b$

**NON-LINEAR REGRESSION**

**SINE**  
 $f(x) = a \sin[b(x+c)] + d$

**CUBIC**  
 $f(x) = ax^3 + bx^2 + cx + d$

**QUADRATIC**  
 $f(x) = ax^2 + bx + c$

**EXPONENTIAL**  
 $f(x) = a \cdot b^x$

GDC  $\Rightarrow a = 0.031, b = 0.591, c = -1.16$   
 $f(x) = 0.031x^2 + 0.591x - 1.16$

x	y	f(x)	y-f(x)	[y-f(x)] <sup>2</sup>
2	1	0.15	+0.85	0.73
5	2	2.57	-0.57	0.32
10	5	7.85	-2.85	8.12
14	17	13.19	+3.81	14.52
19	20	21.26	-1.26	1.59

$SS_{RES} = \sum_{i=1}^5 [y_i - f(x_i)]^2 = 25.3$

GDC  $\Rightarrow R^2 = 0.919$  THE COEFFICIENT OF DETERMINATION  
 $R^2 = 1 - \frac{SS_{RES}}{SS_{TOT}}$   
 THE PROPORTION OF VARIABILITY IN  $y$  ACCOUNTED FOR BY  $f$ .  
 $R^2 = 1 \Leftrightarrow SS_{RES} = 0$

**$X \sim \text{Po}(\lambda)$**   
 $P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}, x=0,1,2,\dots$   
 $E(X) = \lambda$   
 $\text{Var}(X) = \lambda$

**POISSON DISTRIBUTION**

$X \sim \text{Po}(\lambda_1)$   
 $Y \sim \text{Po}(\lambda_2)$   
 $\Rightarrow X+Y \sim \text{Po}(\lambda_1 + \lambda_2)$

INDEPENDENT  $x$  EVENTS  
 UNIFORM AVERAGE RATE  $\lambda$   
 CONTINUOUS INTERVAL

**PARALLEL FORMS**  
 RELIABILITY TESTS  
 TEST-RETEST

**SURVEY DESIGN**  
 DATA COLLECTION  
 CHOOSE RELEVANT DATA AND VARIABLES

**CONTENT**  
 VALIDITY TESTS  
 CRITERION RELATED

$\bar{x}$  IS AN UNBIASED ESTIMATOR OF  $\mu$

$S_{n-1}^2 = \sum_{i=1}^n \frac{(x_i - \bar{x})^2}{n-1}$   
 IS AN UNBIASED ESTIMATOR OF  $\sigma^2$

POPULATION  $\mu, \sigma^2$   
 SAMPLE  $\bar{x}, S_n^2$

**COMBINING RANDOM VARIABLES**

$Y = a_1X_1 + a_2X_2 + \dots + a_nX_n$   
 $E(Y) = a_1E(X_1) + a_2E(X_2) + \dots + a_nE(X_n)$   
 $\text{Var}(Y) = a_1^2 \text{Var}(X_1) + a_2^2 \text{Var}(X_2) + \dots + a_n^2 \text{Var}(X_n)$   
 $X_i$  INDEPENDENT

**CENTRAL LIMIT THEOREM**  
 For  $n > 30$   $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

**CONFIDENCE INTERVALS**

$\bar{x} - z \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z \frac{\sigma}{\sqrt{n}}$  ( $\sigma$  IS KNOWN)  
 $\bar{x} - t \frac{S_{n-1}}{\sqrt{n}} < \mu < \bar{x} + t \frac{S_{n-1}}{\sqrt{n}}$  ( $\sigma$  IS UNKNOWN)

**MARKOV CHAINS**

INITIAL STATE  $\Rightarrow x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  START AT (A)

STEADY STATE  $\Rightarrow x^T = \text{NO CHANGE}$

$\begin{pmatrix} 0.2 & 0.6 & 0 \\ 0.5 & 0 & 0.7 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\Rightarrow 0.2x + 0.6y + 0z = x$   
 $\Rightarrow 0.5x + 0y + 0.7z = y$   
 $\Rightarrow 0.3x + 0.4y + 0.3z = z$

$\Rightarrow -0.8x + 0.6y = 0$   
 $\Rightarrow 0.5x - y + 0.7z = 0$   
 $\Rightarrow 0.3x + 0.4y - 0.7z = 0$

AND  $x + y + z = 1$

GDC SOLVER  
 $\Rightarrow x = 0.284$   
 $y = 0.378$   
 $z = 0.338$

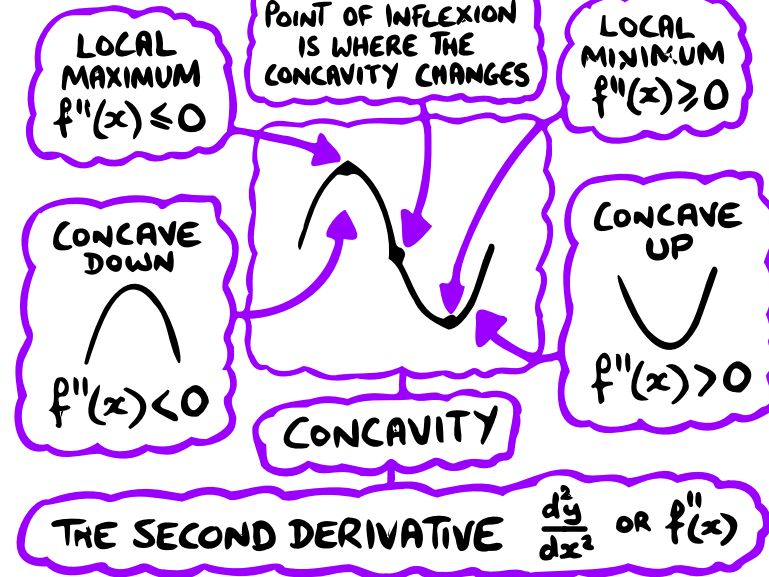
AFTER  $n$  MOVES  $\Rightarrow x_n = T^n x_0$

e.g. AFTER 3 MOVES  $\Rightarrow x_3 = \begin{pmatrix} 0.2 & 0.6 & 0 \\ 0.5 & 0 & 0.7 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}^3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

STEADY STATE AS  $n \rightarrow \infty \Rightarrow T^n x_0 \rightarrow \begin{pmatrix} 0.284 \\ 0.378 \\ 0.338 \end{pmatrix}$

AIR IS PUMPED INTO A SPHERICAL BALLOON AT A RATE OF  $8 \text{ cm}^3/\text{sec}$ . DETERMINE THE RATE AT WHICH THE RADIUS IS INCREASING WHEN THE DIAMETER IS 10 cm.

SPHERE  $\Rightarrow V = \frac{4}{3}\pi r^3$   
 $\therefore \frac{dV}{dr} = 4\pi r^2$   
 $\frac{dV}{dt} = \frac{dr}{dt} \times \frac{dV}{dr} = \frac{8}{4\pi r^2}$   
 $\therefore$  WHEN  $r = 5 \text{ cm} \Rightarrow \frac{dr}{dt} = \frac{8}{4\pi \cdot 5^2} = \frac{2}{25\pi} \text{ cm/s}$



e.g.  $y = x^2 \sin x$   
 $\frac{dy}{dx} = x^2 \cos x + 2x \sin x$

e.g.  $y = \sin(x^2) \Rightarrow \frac{dy}{dx} = 2x \cdot \cos(x^2)$   
 CHAIN RULE

PRODUCT RULE  
 $y = uv$   
 $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

CHAIN RULE  
 $y = g(u), u = f(x)$   
 $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

QUOTIENT RULE  
 $y = \frac{u}{v}$   
 $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$\frac{dy}{dx} = x + y$ . WHEN  $x=0, y=1$  USING  $h=0.2$  ESTIMATE  $y$  WHEN  $x=1$

n	$x_n$	$y_n \Rightarrow$ STORE 0.2 $\rightarrow h$ , USE ANS ON GDC
0	0	1
1	0.2	= ANS + h(0 + ANS) = 1.2
2	0.4	= ANS + h(0.2 + ANS) = 1.48
3	0.6	= ANS + h(0.4 + ANS) = 1.856
4	0.8	= ANS + h(0.6 + ANS) = 2.347
5	1	= ANS + h(0.8 + ANS) = 2.977

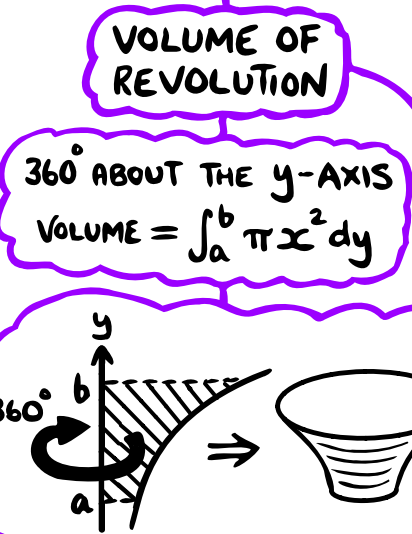
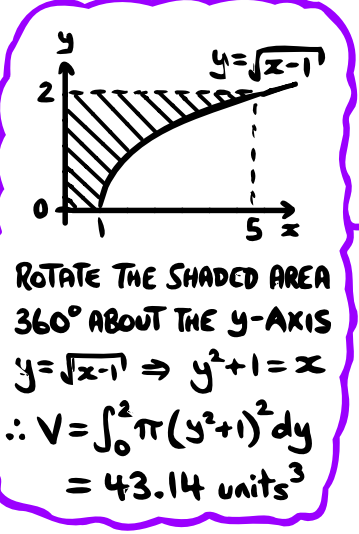
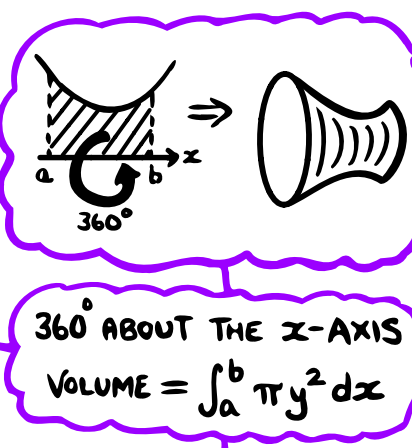
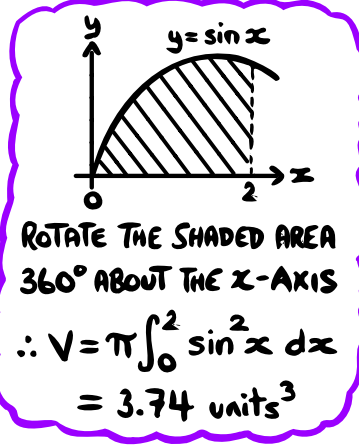
$x_5 = 1, y_5 = 2.977$

$\frac{dx}{dt} = 3x - y, \frac{dy}{dt} = x + y + t$ . WHEN  $t=0, x=2, y=1$  USING A STEP OF  $h = \frac{1}{4}$ , ESTIMATE  $x$  AND  $y$  WHEN  $t=1$

t	x	y
0	$x_0 = 2$	$y_0 = 1$
0.25	$x_1 = 2 + (3 \times 2 - 1)(\frac{1}{4}) = 3.25$	$y_1 = 1 + (2 + 1 + 0)(\frac{1}{4}) = 1.75$
0.5	$x_2 = 3.25 + (3 \times 3.25 - 1.75)(\frac{1}{4}) = 5.25$	$y_2 = 1.75 + (3.25 + 1.75 + 0.25)(\frac{1}{4}) = 3.063$
0.75	$x_3 = 5.25 + (3 \times 5.25 - 3.063)(\frac{1}{4}) = 8.422$	$y_3 = 3.063 + (5.25 + 3.063 + 0.5)(\frac{1}{4}) = 5.266$
1	$x_4 = 8.422 + (3 \times 8.422 - 5.266)(\frac{1}{4}) = 13.422$	$y_4 = 5.266 + (8.422 + 5.266 + 0.75)(\frac{1}{4}) = 8.875$

$\therefore$  WHEN  $t=1 \Rightarrow x = 13.422, y = 8.875$

RELATED RATES OF CHANGE



FURTHER DIFFERENTIATION

$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$   
 $\int e^x \, dx = e^x + c$   
 $\int \frac{1}{x} \, dx = \ln|x| + c$   
 $\int \sin x \, dx = -\cos x + c$   
 $\int \cos x \, dx = \sin x + c$   
 $\int \frac{1}{\cos^2 x} \, dx = \tan x + c$

FURTHER INTEGRATION

REVERSE CHAIN RULE OF THE FORM  $\int g(x)f(j(x)) \, dx$   
 REPLACE  $x \rightarrow ax+b$   
 $\int 2x(x^2+1)^4 \, dx = \frac{1}{5}(x^2+1)^5 + c$   
 $\int e^{3x+2} \, dx = \frac{1}{3}e^{3x+2} + c$   
 $\int \cos(2x-1) \, dx = \frac{1}{2}\sin(2x-1) + c$

KINEMATICS

SPEED =  $|v|$   
 $\frac{d}{dt} S = \text{DISPLACEMENT} \Rightarrow \int dt$   
 $\frac{d}{dt} v = \text{VELOCITY} \Rightarrow \int dt$   
 $\frac{d}{dt} a = \text{ACCELERATION} \Rightarrow \int dt$   
 DISPLACEMENT FROM A TO D =  $\int v \, dt = 10 - 12 + 7 = 5 \text{ m}$   
 DIST. TRAVELLED =  $\int |v| \, dt = 10 + 12 + 7 = 29 \text{ m}$

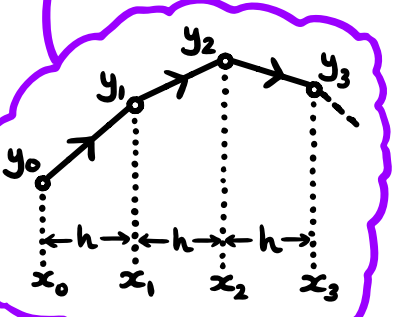
DIFFERENTIAL EQUATIONS

VARIABLES SEPARABLE  $\frac{dy}{dx} = f(x)g(y)$   
 SEPARATE AND INTEGRATE  $\int \frac{1}{g(y)} \, dy = \int f(x) \, dx$   
 $\frac{dy}{dx} = 3y$  AND  $y=2$  WHEN  $x=0$   
 $\Rightarrow \int \frac{1}{y} \, dy = \int 3 \, dx$   
 $\Rightarrow \ln|y| = 3x + c$   
 $\Rightarrow y = e^{3x+c} = e^{3x} \cdot e^c = Ae^{3x}$   
 $(0,2) \Rightarrow 2 = Ae^{3(0)} \Rightarrow A=2$   
 $\therefore y = 2e^{3x}$

THE GROWTH OF AN ALGAE  $G$  AT TIME  $t$  IS PROPORTIONAL TO  $\sqrt{G}$ . INITIALLY ALGAE COVERS  $9 \text{ m}^2$  OF THE SURFACE OF A POND AND 4 DAYS LATER THIS INCREASES TO  $25 \text{ m}^2$ . WHAT WILL THE AREA OF ALGAE BE AFTER 1 WEEK?

$\frac{dG}{dt} = k\sqrt{G} \Rightarrow \int \frac{1}{\sqrt{G}} \, dG = \int k \, dt$   
 $\therefore 2G^{1/2} = kt + c$   
 $t=0, G=9 \Rightarrow c=6$   
 $\therefore 2G^{1/2} = kt + 6$   
 $t=4, G=25 \Rightarrow k=1$   
 $\therefore G = (\frac{t+6}{2})^2$   
 $t=7 \Rightarrow G = (\frac{7+6}{2})^2 = 42.25 \text{ m}^2$

$y_{n+1} = y_n + hf(x_n, y_n)$   
 $x_{n+1} = x_n + h$



FIRST ORDER

EULER'S METHOD

NUMERICAL SOLUTIONS

$\frac{dx}{dt} = f_1(x, y, t)$   
 $\frac{dy}{dt} = f_2(x, y, t)$   
 $x_{n+1} = x_n + f_1(x_n, y_n, t_n)h$   
 $y_{n+1} = y_n + f_2(x_n, y_n, t_n)h$

COUPLED SYSTEM

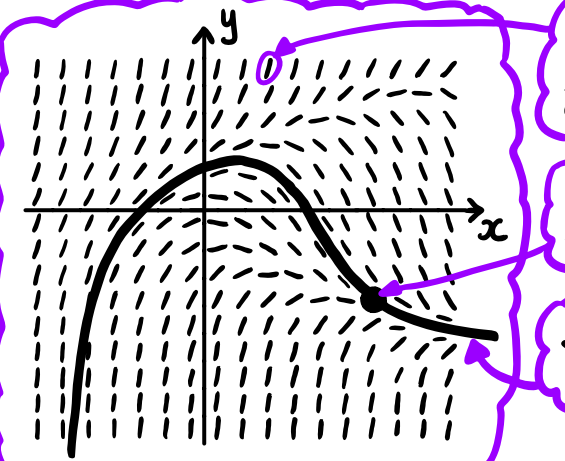
SECOND ORDER

$\frac{d^2x}{dt^2} = f(x, \frac{dx}{dt}, t)$

LET  $y = \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{d^2x}{dt^2}$   
 NOW SOLVE  $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = f(x, y, t) \end{cases}$

SLOPE FIELDS

e.g.  $\frac{dy}{dx} = y^2 - x$



DIFFERENTIAL EQUATIONS

$\frac{dx}{dt} = ax + by$   
 $\frac{dy}{dt} = cx + dy$

BOUNDARY CONDITION

SOLUTION CURVE

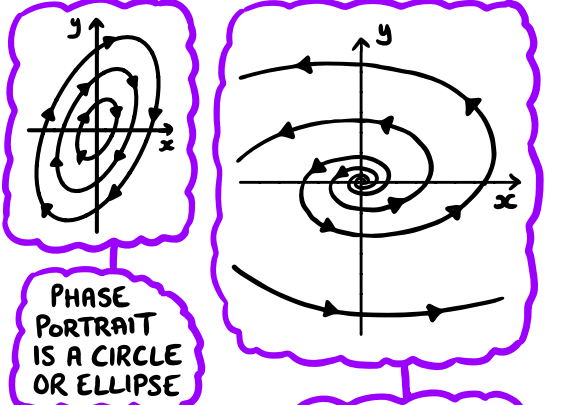
AT (1,2)  $\frac{dy}{dx} = 2^2 - 1 = 3$

$(x', y') = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow x' = Ax$

EIGENVECTORS

$\lambda_1 = 4 \Rightarrow z_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$   
 $\lambda_2 = -1 \Rightarrow z_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

FIND THE EIGENVALUES OF  $A \Rightarrow \lambda_1, \lambda_2$



PHASE PORTRAIT IS A CIRCLE OR ELLIPSE

IMAGINARY  $\lambda_1$  AND  $\lambda_2$

PHASE PORTRAIT IS A SPIRAL

COMPLEX  $\lambda_1$  AND  $\lambda_2$

REAL PART OF BOTH  $\lambda_1$  AND  $\lambda_2$  IS -

SOLUTIONS MOVE TOWARDS THE ORIGIN

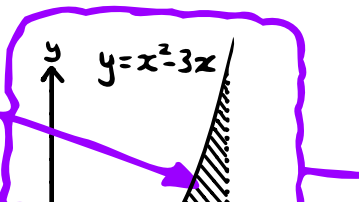
REAL PART OF BOTH  $\lambda_1$  AND  $\lambda_2$  IS +

SOLUTIONS MOVE AWAY FROM ORIGIN

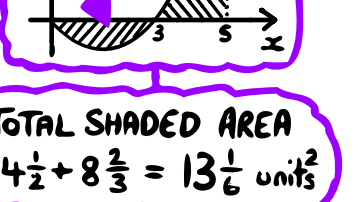
REAL WITH  $\lambda_1 = +$  AND  $\lambda_2 = -$

PHASE PORTRAIT IS A SADDLE

$\int_3^5 x^2 - 3x \, dx = [\frac{1}{3}x^3 - \frac{3}{2}x^2]_3^5 = (\frac{125}{3} - \frac{45}{2}) - (\frac{27}{3} - \frac{27}{2}) = 8\frac{2}{3}$



$\int_0^3 x^2 - 3x \, dx = [\frac{1}{3}x^3 - \frac{3}{2}x^2]_0^3 = (\frac{27}{3} - \frac{27}{2}) - (0 - 0) = -4\frac{1}{2}$



DEFINITE INTEGRATION TO FIND AREA  
 $\int_a^b g(x) \, dx = g(b) - g(a)$

TOTAL SHADED AREA =  $4\frac{1}{2} + 8\frac{2}{3} = 13\frac{1}{6} \text{ units}^2$