

**MATHEMATICAL METHODS
STANDARD LEVEL
PAPER 2**

Wednesday 5 November 2003 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.
- Write the make and model of your calculator in the appropriate box on your cover sheet
e.g. Casio *fx-9750G*, Sharp EL-9600, Texas Instruments TI-85.

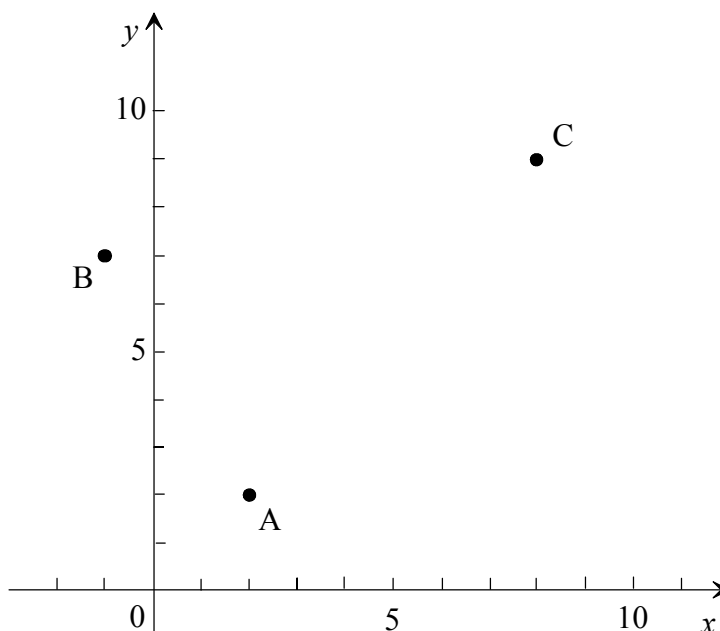
Please start each question on a new page. You are advised to show all working, where possible. Where an answer is wrong, some marks may be given for correct method, provided this is shown by written working. Solutions found from a graphic display calculator should be supported by suitable working e.g. if graphs are used to find a solution, you should sketch these as part of your answer.

SECTION A

Answer all **five** questions from this section.

1. [Maximum mark: 16]

The diagram shows points A, B and C which are three vertices of a parallelogram ABCD. The point A has position vector $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.



- (a) Write down the position vector of B and of C. [2 marks]
- (b) The position vector of point D is $\begin{pmatrix} d \\ 4 \end{pmatrix}$. Find d . [3 marks]
- (c) Find \vec{BD} . [1 mark]

(This question continues on the following page)

(Question 1 continued)

The line L passes through B and D.

- (d) (i) Write down a vector equation of L in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix} + t \begin{pmatrix} m \\ n \end{pmatrix}.$$

- (ii) Find the value of t at point B.

[3 marks]

- (e) Let P be the point (7, 5). By finding the value of t at P, show that P lies on the line L .

[3 marks]

- (f) Show that \vec{CP} is perpendicular to \vec{BD} .

[4 marks]

2. [Maximum mark: 13]

The diagram below shows a circle, centre O, with a radius 12 cm. The chord AB subtends at an angle of 75° at the centre. The tangents to the circle at A and at B meet at P.

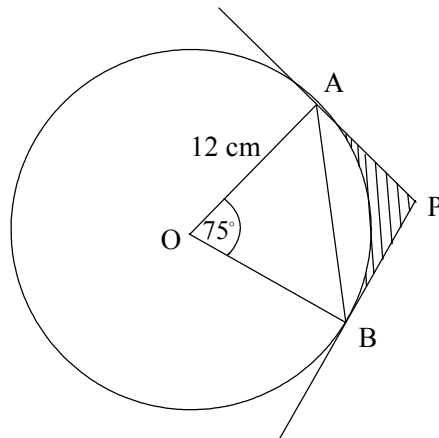


diagram not to scale

- (a) Using the cosine rule, show that the length of AB is $12\sqrt{2(1 - \cos 75^\circ)}$.

[2 marks]

- (b) Find the length of BP.

[3 marks]

- (c) Hence find

- (i) the area of triangle OBP;

- (ii) the area of triangle ABP.

[4 marks]

- (d) Find the area of **sector** OAB.

[2 marks]

- (e) Find the area of the shaded region.

[2 marks]

3. [Maximum mark: 11]

The diagrams below show the first four squares in a sequence of squares which are subdivided in half. The area of the shaded square A is $\frac{1}{4}$.

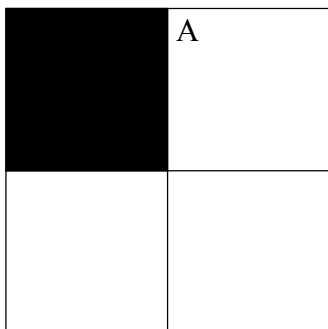


Diagram 1

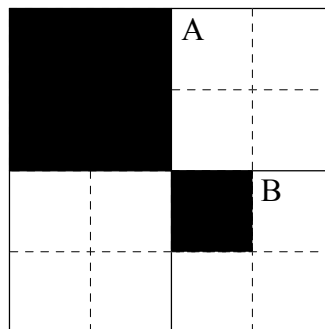


Diagram 2

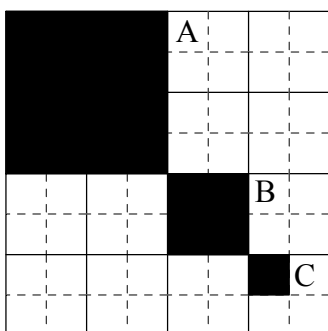


Diagram 3

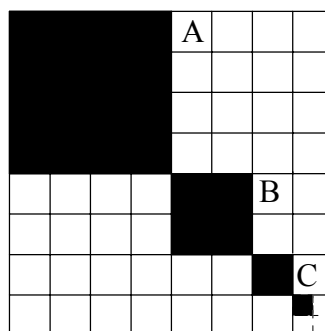


Diagram 4

- (a) (i) Find the area of square B and of square C.
- (ii) Show that the areas of squares A, B and C are in geometric progression.
- (iii) Write down the common ratio of the progression. [5 marks]
- (b) (i) Find the **total** area shaded in diagram 2.
- (ii) Find the **total** area shaded in the 8th diagram of this sequence. Give your answer correct to **six** significant figures. [4 marks]
- (c) The dividing and shading process illustrated is continued indefinitely. Find the total area shaded. [2 marks]

4. [Maximum mark: 14]

Consider the function $f(x) = 1 + e^{-2x}$.

(a) (i) Find $f'(x)$.

(ii) Explain briefly how this shows that $f(x)$ is a decreasing function for all values of x (*i.e.* that $f(x)$ always decreases in value as x increases).

[2 marks]

Let P be the point on the graph of f where $x = -\frac{1}{2}$.

(b) Find an expression in terms of e for

(i) the y -coordinate of P;

(ii) the gradient of the tangent to the curve at P.

[2 marks]

(c) Find the equation of the tangent to the curve at P, giving your answer in the form $y = ax + b$.

[3 marks]

(d) (i) Sketch the curve of f for $-1 \leq x \leq 2$.

(ii) Draw the tangent at $x = -\frac{1}{2}$.

(iii) Shade the area enclosed by the curve, the tangent and the y -axis.

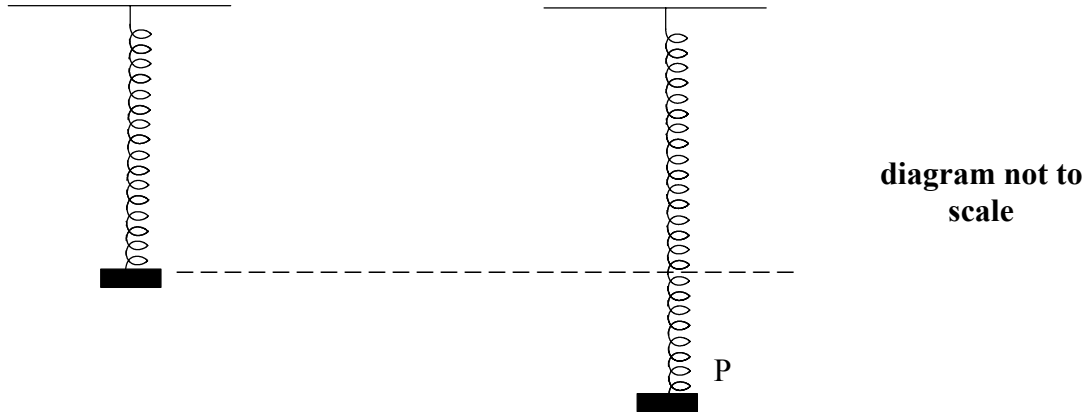
(iv) Find this area.

[7 marks]

5. [Maximum mark: 16]

Note: Radians are used throughout this question.

A mass is suspended from the ceiling on a spring. It is pulled down to point P and then released. It oscillates up and down.



Its distance, s cm, from the ceiling, is modelled by the function $s = 48 + 10\cos 2\pi t$ where t is the time in seconds from release.

- (a) (i) What is the distance of the point P from the ceiling?
- (ii) How long is it until the mass is next at P? [5 marks]
- (b) (i) Find $\frac{ds}{dt}$.
- (ii) Where is the mass when the velocity is zero? [7 marks]

A second mass is suspended on another spring. Its distance r cm from the ceiling is modelled by the function $r = 60 + 15\cos 4\pi t$. The two masses are released at the same instant.

- (c) Find the value of t when they are first at the same distance below the ceiling. [2 marks]
- (d) In the first three seconds, how many times are the two masses at the same height? [2 marks]

SECTION B

Answer **one** question from this section.

Statistical Methods

6. [Maximum mark: 30]

- (i) It is claimed that the masses of a population of lions are normally distributed with a mean mass of 310 kg and a standard deviation of 30 kg.

- (a) Calculate the probability that a lion selected at random will have a mass of 350 kg or more.

[2 marks]

- (b) The probability that the mass of a lion lies between a and b is 0.95, where a and b are symmetric about the mean. Find the value of a and of b .

[3 marks]

A biologist wishes to test the claim that the mean mass is 310 kg. A random sample of 36 lions is studied. The mean mass of this sample is 300 kg. The population standard deviation of 30 kg is assumed to be correct.

- (c) Calculate the probability of obtaining a sample mean of 300 kg or less if the population mean of 310 kg is correct.

[3 marks]

- (d) The biologist concludes the population mean is less than 310 kg. Justify this conclusion.

[2 marks]

- (ii) It is claimed that girls are better than boys in geography. To test this hypothesis, a test is given to 120 girls and 80 boys. Students either pass or fail the test. The outcome is as follows

	Pass	Fail	Total
Girls	96	24	120
Boys	54	26	80
Total	150	50	200

- (a) Construct a table of expected values for the numbers passing and failing, assuming ability in geography is independent of gender.

[4 marks]

- (b) Calculate χ^2 for this data. (The use of Yates' correction is not required.)

[2 marks]

- (c) The critical value of χ^2 at the 5 % level of significance is 3.841. What is the conclusion?

[1 mark]

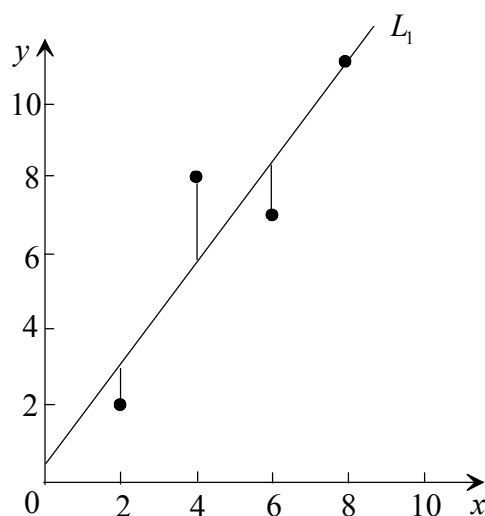
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(Question 6 continued)

- (iii) An experiment gives the following **four** data points (x, y)

$$(2, 2), (4, 8), (6, 7), (8, 11).$$

The least squares regression line L_1 of y on x has the equation $y = 1.3x + 0.5$. The scatter diagram below is a sketch of the four points and the line L_1 .



The vertical distances between the data points and the line L_1 are given in the following table.

Point	Distance
$(2, 2)$	1.1
$(4, 8)$	c
$(6, 7)$	d
$(8, 11)$	0.1

- (a) (i) Show that $c = 2.3$ and calculate d .
- (ii) Calculate $V = 1.1^2 + c^2 + d^2 + 0.1^2$.
- (iii) Another straight line (not the least squares regression line) is drawn on the diagram. The new vertical distances (from the four points to this line) are calculated. How does the sum of the squares of these distances compare to V ?

[5 marks]

(This question continues on the following page)

(Question 6 continued)

- (b) The sum of the squares of the horizontal distances of the four points from L_1 is 4.85.

Let H be the sum of the squares of the horizontal distances of the four points from any other straight line.

Which of the following statements is true?

- (i) $H \leq 4.85$ for **all** other lines.
- (ii) $H \geq 4.85$ for **all** other lines.
- (iii) $H \leq 4.85$ for some lines and $H \geq 4.85$ for other lines.

[2 marks]

- (c) The x and y coordinates of the four points are interchanged, giving the new points $(2, 2)$, $(8, 4)$, $(7, 6)$, and $(11, 8)$. Find the equation of the least squares regression line L_2 for these new points.

[3 marks]

- (d) It is noted that $\frac{2+4+6+8}{4} = 5$. The point $(5, 7)$ is on L_1 and $(7, 5)$ is on L_2 .

What property of the least squares regression line does this statement illustrate?

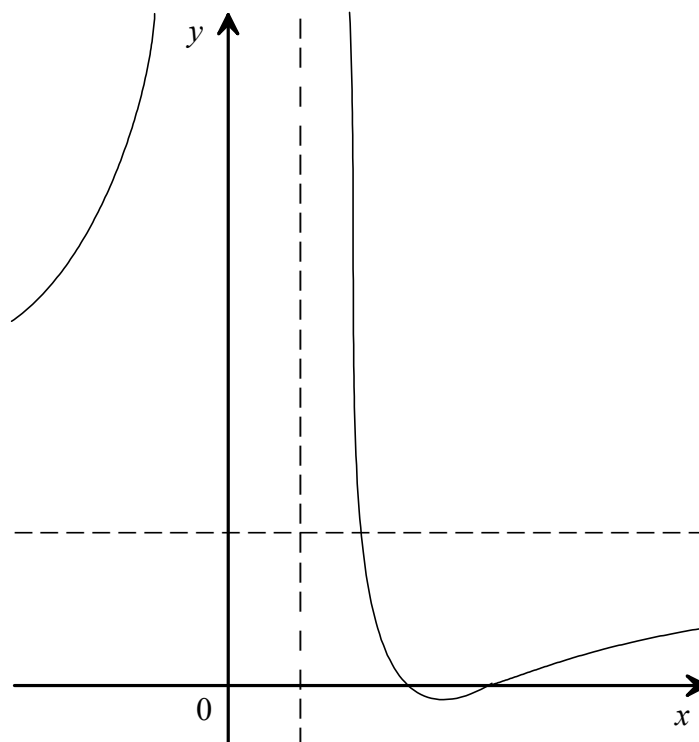
[3 marks]

Further Calculus

7. [Maximum mark: 30]

- (i) Consider the function f given by $f(x) = \frac{2x^2 - 13x + 20}{(x-1)^2}$, $x \neq 1$.

A part of the graph of f is given below.



(This question continues on the following page)

(Question 7 continued)

The graph has a vertical asymptote and a horizontal asymptote, as shown.

- (a) Write down the **equation** of the vertical asymptote. *[1 mark]*
- (b) $f(100) = 1.91$, $f(-100) = 2.09$, $f(1000) = 1.99$
- (i) Evaluate $f(-1000)$.
- (ii) Write down the **equation** of the horizontal asymptote. *[2 marks]*
- (c) Show that $f'(x) = \frac{9x-27}{(x-1)^3}$, $x \neq 1$. *[3 marks]*

The second derivative is given by $f''(x) = \frac{72-18x}{(x-1)^4}$, $x \neq 1$.

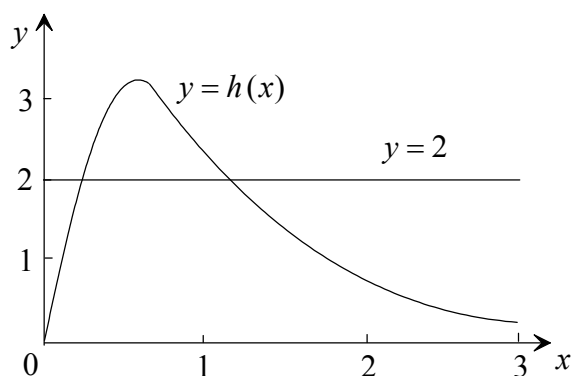
- (d) Using values of $f'(x)$ and $f''(x)$, explain why a minimum must occur at $x = 3$. *[2 marks]*
- (e) There is a point of inflexion on the graph of f . Write down the coordinates of this point. *[2 marks]*

(This question continues on the following page)

(Question 7 continued)

- (ii) Consider the function $h : x \mapsto \frac{10x}{(x^2 + 1)^2}$.

The graph of this function for $0 \leq x \leq 3$ is given below.



The value of the function is 2 at a point near $x = 0.2$, and at another point near $x = 1.2$.

- (a) Show that the equation $h(x) = 2$ may be rewritten in the form

$$x = \frac{(x^2 + 1)^2}{5}. \quad [1 \text{ mark}]$$

- (b) Define $g(x) = \frac{(x^2 + 1)^2}{5}$. Use fixed-point iteration with $x_0 = 0.2$ and $x_{n+1} = g(x_n)$ to find

(i) x_1 ;

- (ii) the solution to $h(x) = 2$ near $x = 0.2$. [3 marks]

Give your answers to **both** parts to an accuracy of **five** significant figures.

(This question continues on the following page)

(Question 7 continued)

(c) (i) Find $g'(x)$.

(ii) Evaluate

(a) $g'(0.2)$;

(b) $g'(1.2)$.

(iii) Explain why fixed-point iteration using $g(x)$ will not converge to the solution near $x = 1.2$, no matter what value of x_0 is used.

[6 marks]

(d) Using any other method, find the solution near $x = 1.2$, to an accuracy of **six** significant figures.

[2 marks]

(e) (i) Using the substitution $u = x^2 + 1$, find $\int \frac{10x}{(x^2 + 1)^2} dx$.

(ii) Given that $\int_0^a \frac{10x}{(x^2 + 1)^2} dx = 4$, where $a > 0$, find a .

[8 marks]

Further Geometry

8. [Maximum mark: 30]

(i) Consider the matrix $A = \begin{pmatrix} 5 & -2 \\ 7 & 1 \end{pmatrix}$.

(a) Write down the inverse, A^{-1} .

[2 marks]

(b) B , C and X are also 2×2 matrices.

(i) Given that $XA + B = C$, express X in terms of A^{-1} , B and C .

(ii) Given that $B = \begin{pmatrix} 6 & 7 \\ 5 & -2 \end{pmatrix}$, and $C = \begin{pmatrix} -5 & 0 \\ -8 & 7 \end{pmatrix}$, find X .

[4 marks]

(ii) The matrix $M = \begin{pmatrix} 0.6 & 0.2 \\ -0.8 & 1.4 \end{pmatrix}$ represents a transformation.

(a) Calculate the area scale factor for this transformation.

[2 marks]

Consider the square with vertices at $O(0, 0)$, $A(1, 2)$, $B(-1, 3)$ and $C(-2, 1)$.

(b) The images under M of O , A , B and C are O' , A' , B' and C' respectively. The coordinates of O' are $(0, 0)$. The coordinates of B' are $(0, 5)$.

(i) Find the coordinates of

(a) A' ;

(b) C' .

(ii) Sketch a diagram showing the square $OABC$, and the parallelogram $O'A'B'C'$.

(iii) **Hence**, give a complete geometric description of M .

[8 marks]

(This question continues on the following page)

(Question 8 continued)

(c) The line with equation $y = 2x$ makes an angle θ with the x -axis.

(i) Show that $\cos \theta = \frac{1}{\sqrt{5}}$.

(ii) Find a similar expression for $\sin \theta$. [3 marks]

(d) Let $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$.

(i) Give a complete geometric description of the transformations represented by

(a) \mathbf{P} ;

(b) \mathbf{Q} ;

(c) \mathbf{R} .

(ii) It can be shown that $\mathbf{PQR} = \mathbf{M}$, so that \mathbf{M} is a composite of transformations \mathbf{P} , \mathbf{Q} and \mathbf{R} . Which transformation is applied first? [7 marks]

(iii) The transformation \mathbf{T} is defined by

$$\mathbf{T}: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4.4 \\ 8.8 \end{pmatrix}.$$

It represents a reflection in a straight line which does **not** pass through the origin. Find the equation of this line in the form $ax + by + c = 0$ where $a, b, c \in \mathbb{Z}$. [4 marks]